UNIVERSITY OF CALIFORNIA
Santa Barbara

The Moduli Space of Flat Connections over Higher Dimensional Manifolds

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Mathematics

by

Casey Alexander Blacker

Committee in Charge:

Professor Xianzhe Dai, Chair
Professor John Douglas Moore
Professor Guofang Wei

June 2018

The dissertation of Casey Alexander Blacker is approved.

Professor John Douglas Moore

Professor Guofang Wei

Professor Xianzhe Dai, Committee Chairperson

June 2018

The Moduli Space of Flat Connections over Higher Dimensional Manifolds

Casey Alexander Blacker

Acknowledgements

I am grateful to my advisor, Xianzhe Dai, for his guidance and assistance on this project, and to the rest of my advisory committee, Doug Moore and Guofang Wei, for their helpful comments and feedback.

I would also like to extend my gratitude to Eugene Lerman and Tatyana Barron for their stimulating discussions and helpful suggestions regarding the vector-valued symplectic formalism.

Special thanks go to Alex Kinsella and Matt Brown of the Department of Physics, whose thorough and enthusiastic explanations of the modern physical implications of the relevant background material are greatly appreciated. I thank my graduate and postdoctoral colleagues in the geometry group, Yihan Li, Joey Chahine, Zhongmin Jin, and Shoo Seto for their feedback and support.

I also wish to acknowledge my appreciation for the mentorship and assistance of Lars Olsen, Collin Bleak, and James Mitchel during my years as an undergraduate.

Finally, I would like to thank my family for their continual support throughout my education.

For my parents.

Curriculum Vitæ

Education

2018 Ph.D. in Mathematics (Expected), University of California, Santa Barbara
with Certificate in College and University Teaching

2015 M.A. in Mathematics, University of California, Santa Barbara

2013 Master of Mathematics (Pure Mathematics, Fast-Track), University of St Andrews, Scotland

Publications

2018 *First eigenvalue of the p-Laplacian on Kähler manifolds.* (With Shoo Seto.) Submitted.

Invited Talks

2018 *Vector-Valued Symplectic Geometry and the Moduli Space of Flat Connections.* Math Connections, University of California, Riverside

2018 *The Moduli Space of Flat Connections Over Higher Dimensional Manifolds.* Joint Mathematics Meetings, San Diego

Honors and Awards

1. RTG in Geometry and Topology, Ph.D. Research Fellowship 2013-2017

Abstract

The Moduli Space of Flat Connections over Higher Dimensional Manifolds

by

Casey Alexander Blacker

Let M be a smooth manifold of dimension at least 3, let G be a compact Lie group, and let P be a G-principal bundle on M. This work is motivated by two aims:

1. Exhibit the moduli space $\mathcal{M}(P)$ of flat connections on P as a generalized symplectic reduction of the space $\mathcal{A}(P)$ of connection on P by the action of the gauge group $\mathcal{G}$.

2. Compute the symplectic volume of the moduli space $\mathcal{M}_G(M)$ of all flat G-connections on M.

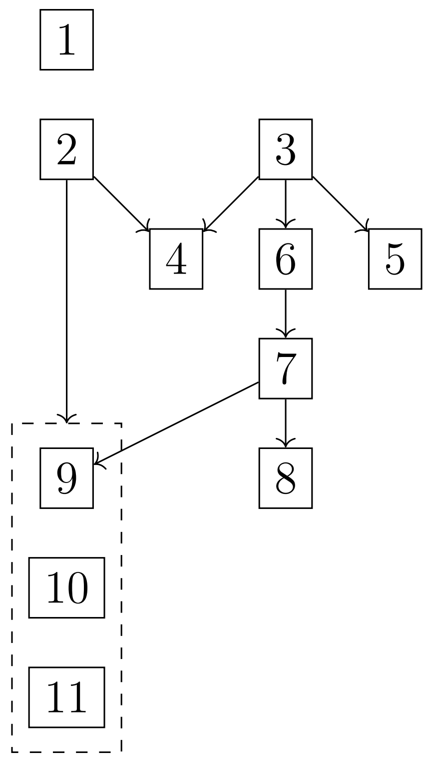
We show that the appropriate adaptation of the Hamiltonian formalism in this context is to consider a natural $\Omega^2(M)/B^2(M)$-valued symplectic form ω on $\mathcal{A}(P)$. With The action of the gauge group $\mathcal{G}$ on the space of connections $(\mathcal{A}, \omega)$ admits a natural moment map μ, and the reduction of the vector-valued Hamiltonian system $(\mathcal{A}, \omega, \mathcal{G}, \mu)$ is the moduli space of flat connections $\mathcal{M}$. The reduced form ω_0, which may be nondegenerate, takes values in the second cohomology $H^2(M)$ of the underlying manifold M.

Several chapters are devoted to the theory of vector-valued symplectic geometry. In addition to its applications to the moduli space of flat connections, we show that the vector-valued symplectic formalism has a rich structure that does not always reflect its real-valued counterpart. We prove two symplectic reduction theorems and investigate the vector-valued analogues of Hamiltonian and Lagrangian mechanics.

We also compute the volume of the moduli space $\mathcal{M}(M)$ of all G-connections on a symplectic manifold (M, ω) in two special cases. First, we assume that the structure group G is abelian; second, that G is semisimple and the fundamental group $\pi_1(M)$ is free abelian. The expression of the volume is a function of the covolume of the lattice $H^1(M, \mathbb{Z})$ in $H^1(M, \mathbb{R})$, the rank of the homology $H^1(M, \mathbb{R})$, and the structure group G.

Contents

Interdependence of the Chapters



Chapter 1

Introduction

Let Σ be a closed oriented surface, G a Lie group with Ad-invariant metric $\langle\,,\rangle$ on the Lie algebra $\mathfrak{g}$, and P a G-principal bundle on Σ. In their seminal paper [2] on the topology of the Yang-Mills moduli space, Atiyah and Bott observed that the space of connections $\mathcal{A}(P)$ on P possesses a natural symplectic structure $\omega \in \Omega^2\big(\mathcal{A}(P)\big)$, given by

$$\omega_A(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta \qquad (*)$$

for $A \in \mathcal{A}(P)$ and $\alpha, \beta \in \Omega^1(\Sigma, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$, where the wedge product is defined to be the composition

$$\wedge : \Omega^*(\Sigma, \mathfrak{g}) \otimes \Omega^*(\Sigma, \mathfrak{g}) \xrightarrow{\wedge_{\Omega^*(\Sigma)}} \Omega^*(\Sigma, \mathfrak{g} \otimes \mathfrak{g}) \xrightarrow{\langle\,,\rangle_{\mathfrak{g}}} \Omega^*(\Sigma)$$

They further observed that the action of the gauge group $\mathcal{G}$ on $\mathcal{A}(P)$ is Hamiltonian and that, under the natural identification of the dual Lie algebra $\mathfrak{g}^*$ of $\mathcal{G}$ with the space of 2-forms $\Omega^2(\Sigma, \operatorname{ad} P)$, the curvature map

$$F : \mathcal{A}(P) \longrightarrow \Omega^2(\Sigma, \mathrm{ad}P)$$

is a moment map for the induced action of $\mathcal{G}$ on $\mathcal{A}(P)$. Thus, the moduli space $\mathcal{M}(P)$ of flat connections on P, defined to be the space of gauge-equivalence classes of flat connections on P, is exhibited as the symplectic reduction $F^{-1}(0)/\mathcal{G}$ of the space of connections $\mathcal{A}(P)$ by the action of the gauge group $\mathcal{G}$. We refer to Chapters 2 and 4 for further details.

In addition to its value as an elegant characterization, this process reveals a natural symplectic structure $\omega_{\mathcal{M}_G(\Sigma)} \in \Omega^2\big(\mathcal{M}(P)\big)$ on the moduli space $\mathcal{M}_G(\Sigma)$ of flat G-connections over Σ, thus enabling the study of $\mathcal{M}_G(\Sigma)$ by symplectic techniques. For example, as it can be shown that this moduli space is finite-dimensional, $\mathcal{M}_G(\Sigma)$ inherits a natural volume form $\frac{1}{n!}\omega^n \in \Omega^{2n}(\mathcal{M})$. The resulting volume $\operatorname{vol} \mathcal{M}_G(\Sigma)$ is relevant in physics; see Section 1.2 below. Another geometric structure that can arise in the presence of a symplectic form is a prequantum line bundle. The study of this aspect of the moduli space is undertaken in [38, 73, 75, 87], among others.

The aim of this dissertation is to investigate the corresponding situation in which the underlying surface Σ is replaced by an compact manifold M of higher dimension. A number of difficulties arise; foremost among which is the dependence of the construction in Equation $(*)$ on the 2-dimensionality of Σ. Indeed, when applied to a higher-dimensional base M, the integral $(*)$ is identically zero. This apparent impasse is overcome by enlarging the space of coefficients which a symplectic form ω is permitted to take.

Notwithstanding an abundance of generalizations of symplectic geometry, see Chapter 5, the vector-valued symplectic formalism is an original development of this dissertation. We introduce it independently in Part II, without reference to the gauge-theoretic material which served as its motivation. The theory is of independent interest and exhibits a wealth of examples. We return in Chapter 9 to apply this theory to the space of connections in the context of higher-dimensional base spaces.

When the underlying space is itself a symplectic manifold (M^{2n}, η), there is a corresponding real-valued symplectic structure on the space of connections $\mathcal{A}(P)$ of a G-principal bundle P on M. Specifically,

$$\omega_A(\alpha, \beta) = \int_M \alpha \wedge \beta \wedge \eta^{n-1}$$

for $A \in \mathcal{A}(P)$ and $\alpha, \beta \in \Omega^1(\Sigma, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$, up to an appropriate scaling constant. In this case, the assignment

$$F \wedge \eta^{n-1} : \mathcal{A}(P) \longrightarrow \Omega^2(\Sigma, \mathrm{ad}P)$$

is a moment map for the action of the gauge group $\mathcal{G}$ on $\mathcal{A}(P)$ and the reduced space $\mathcal{A}(P)_0$ contains the moduli space $\mathcal{M}(P)$ of flat connections on P. In general the two spaces are not equal. If the base (M, η) is *Lefschetz*, that is, if the map

$$H^1(M, \mathbb{R}) \to H^{2n-1}(M, \mathbb{R})$$

$$\alpha \mapsto \alpha \wedge [\eta]^{n-1}$$

is an isomorphism, then $\mathcal{M}(P)$ is a symplectic submanifold, possibly with singularities, of the reduced space $\mathcal{A}(P)_0$.

In Chapters 10 and A, we study the geometry of $\mathcal{M}_G(M)$ with respect to this symplectic structure. Specifically, in Chapter 10, we determine the symplectic volume of $\mathcal{M}_G(M)$ under special restrictions on the structure group G and the fundamental group $\pi_1(M)$. In Chapter 11, we relate $\mathcal{M}_G(M)$ and $\mathcal{M}_G(\Sigma)$ when Σ is an embedded submanifold of M. We briefly touch on the topic of prequantum line bundles in Section 11.2, though there is certainly more to be said on this matter.

1.1 Statement of Results

The results of this dissertation lie within the intersecting domains of vector-valued symplectic geometry (Part II) and the moduli space of flat connections (Part III). We address each in turn. With the exception of Section 10.1, Parts II and III comprise original scholarship.

1.1.1 Vector-Valued Symplectic Geometry

The fundamental definition of the vector-valued symplectic formalism is as follows.

Definition. 7.1 Let V be a vector space. A *V-valued symplectic manifold*, or *V-symplectic manifold* is a smooth manifold M equipped with a closed 2-form $\omega \in \Omega^2(M, V)$, called a *$V$-valued symplectic structure* or a *V-symplectic structure*, which is nondegenerate in the sense that $\iota_X \omega \neq 0$ for every $X \in \mathfrak{X}(M)$.

A vector-valued symplectic structure is thus a straightforward generalization of the notion of a classical, real-valued symplectic structure. We give one key example here and refer to Chapters 7 and 8 for others.

Example. 7.3 Let G be a semisimple Lie group and let $\theta \in \Omega^1(M, \mathfrak{g})$ denote the Maurer-Cartan form on G. The Maurer-Cartan theorem asserts that $-\mathrm{d}\theta = \theta^*[\,,]$, which is nondegenerate by the semisimplicity of G. Thus $\omega = -\mathrm{d}\theta$ is a $\mathfrak{g}$-valued symplectic form on G. When G is compact and $Y \in \mathfrak{g}$, the Y-component of ω is given by

$$\langle Y, \omega \rangle = 12\, \iota_{\bar{Y}} \chi$$

where $\bar{Y} \in \mathfrak{X}(G)$ is the unique left-invariant extension of Y.

This is a particularly illustrative example as it demonstrates the fact that, unlike a classical symplectic structure, a vector-valued symplectic structure can be exact. To further contrast the classical and vector-valued cases, we note that a compact semisimple Lie group G satisfies $H^2(G, \mathbb{R}) = 0$ and thus cannot admit a classical symplectic

	classical	V-valued	
symplectic form	$\omega \in \Omega^2(M, \mathbb{R})$	$\omega \in \Omega^2(M, V)$	
Hamiltonian v.f.	$f \in C^\infty(M)$	$f \in C^\infty(M, V)$	$\omega(\,\cdot, X_f) = df$
comoment map	$\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$	$\tilde{\mu} : \mathfrak{g} \to C^\infty(M, V)$	
moment map	$\mu : M \to \mathfrak{g}^*$	$\mu : M \to \mathrm{Hom}(\mathfrak{g}, V)$	

Figure 1.1: Classical symplectic quantities and their V-valued counterparts.

form. Nonetheless, a certain modification of this $\mathfrak{g}$-valued symplectic structure $-\mathrm{d}\theta$ is related to the classical symplectic structures on the coadjoint obits of the dual Lie algebra $\mathfrak{g}^*$; see Example 7.11.

There are corresponding notions of Hamiltonian vector fields, comoment maps, and moment maps. We list these constructions alongside their classical counterparts in Figure 1.1.1.

Just as in the classical situation, there is a symplectic reduction theorem. However, to the extent that the symplectic dual operator $\hat{U} \mapsto \hat{U}^\omega$ on the powerset $\mathcal{P}(U)$ of a vector-valued symplectic vector space (U, ω) is not generally an involution, the reduced space $M_0 = \mu^{-1}(0)/G$ is not generally symplectic.

Theorem (Vector-Valued Symplectic Reduction)**.** *7.1 Let (M, ω, G, μ) be a V-valued Hamiltonian system. If G is connected and the* reduced space $M_0 = \mu^{-1}(0)/G$ *is smooth, then there is a unique V-valued 2-form $\omega_0 \in \Omega^2(M_0, V)$ such that*

$$\pi^*\omega_0 = i^*\omega$$

for the inclusion $i : \mu^{-1}(0) \hookrightarrow M$ and projection $\pi : \mu^{-1}(0) \to M_0$. Moreover, ω_0 is *closed and the kernel of* ω_0 *at* $\pi x \in M_0$ *is equal to*

$$\ker_x \omega_0 = \pi_*\big(T_x\mu^{-1}(0)^\omega/\underline{\mathfrak{g}}_x\big) \leq T_{\pi x}M_0$$

The situation improves when we restrict our attention to a special subclass of vector-valued Hamiltonian systems which formally relate to classical mechanics. Two limiting examples are,

(i) the tangent bundle TQ of an underlying configuration manifold Q, equipped with a suitable V-valued Lagrangian $L : TQ \to V$, and

(ii) the homomorphism bundle $\mathrm{Hom}(TQ, V)$, which carries a canonical V-symplectic structure.

These examples generalize Lagrangian and Hamiltonian mechanics, respectively. It is interesting to note that, while an admissible classical Lagrangian $L : TM \to \mathbb{R}$ induces a diffeomorphism $\mathbb{F}L : TM \to T^*M$, a V-valued Lagrangian $L : TQ \to V$ yields at most a symplectic immersion $\mathbb{F}L : TQ \to \mathrm{Hom}(TQ, V)$, which can never be an identification when $\dim V \geq 2$. In fact, the image of $\mathbb{F}L$ can even be a compact submanifold of $\mathrm{Hom}(TQ, V)$, as demonstrated in Example 8.1.

Extending the two examples above is the class of *Darboux manifolds*, so named for the eponymous local structure theorem, which we define as follows.

Definition. 8.3 A V-symplectic manifold (M, ω) is said to be *Darboux* if M is locally symplectomorphic to a smooth subbundle of $\pi : \mathrm{Hom}(TQ, V) \to Q$ for some space Q

For a nontrivial example, observe that the inclusion

$$\begin{aligned} f : G &\to \mathrm{Hom}(TG, \mathfrak{g}) \\ g &\mapsto \theta_g \end{aligned}$$

implies that the $\mathfrak{g}$-valued symplectic manifold $(G, -\mathrm{d}\theta)$, as defined above, is Darboux. Here we recall that G is semisimple and $\theta \in \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form on G.

An important property of the collection of Darboux manifolds is that it is preserved under the symplectic reduction by a certain natural class of group actions.

Theorem (Basic reduction of Darboux manifolds)**.** *8.4 If the V-Hamiltonian system (M', ω', G', μ') is locally equivalent to a Hamiltonian subsystem of $\big(\mathrm{Hom}(TQ, V), \omega, G, \mu\big)$ for a basic action of G and standard moment map μ, then the reduced space (M_0, ω_0) is Darboux. In particular, (M_0, ω_0) is symplectic.*

We refer to Section 8.2 for an explanation of the terminology used in this result.

1.1.2 The Moduli Space of Flat Connections

Recall the canonical symplectic structure on the space of connections $\mathcal{A}(P)$ for a G-principal bundle P over a surface Σ,

$$\omega_A(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta,$$

for $A \in \mathcal{A}(P_\Sigma)$ and $\alpha, \beta \in \Omega^1(\Sigma, \mathrm{ad}P_\Sigma) \cong T_A\mathcal{A}(P_\Sigma)$. The catalyst for vector-valued symplectic geometry was the goal of constructing a corresponding symplectic structure when the underlying space Σ is replaced by a manifold M of dimension at least 3, as no such classical symplectic form exists.

Define the 2-form $\omega \in \Omega^2\big(\mathcal{A}(P), \Omega^2(M)/B^2(M)\big)$ to be

$$\omega_A(\alpha, \beta) = \big[\alpha \wedge \beta\big]_{\Omega^2/B^2} \in \Omega^2(M)/B^2(M)$$

for $A \in \mathcal{A}(P)$ and $\alpha, \beta \in \Omega^1(M, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$. Here, $B^2(M) \leq \Omega^2(M)$ denotes the space of 2-coboundaries on M and, as above, the wedge product is defined to be the composition

$$\wedge : \Omega^*(M, \mathfrak{g}) \otimes \Omega^*(M, \mathfrak{g}) \xrightarrow{\wedge_{\Omega^*(M)}} \Omega^*(M, \mathfrak{g} \otimes \mathfrak{g}) \xrightarrow{\langle , \rangle_{\mathfrak{g}}} \Omega^*(M)$$

We first show that this 2-form is a vector-valued symplectic structure on $\mathcal{A}(P)$.

Theorem. *(9.2) The form ω is a $\Omega^2(M)/B^2(M)$-valued symplectic structure on $\mathcal{A}(P)$ if and only if $\dim M = 0$, $\dim M \geq 3$, or M is a closed compact orientable surface.*

The main result of Chapter 9 is that the reduction of $\mathcal{A}(P)$ by the action of the gauge group $\mathcal{G}$ is the moduli space $\mathcal{M}(P)$ of flat connections on P.

Theorem. *(9.3) Let M be either a smooth manifold of dimension at least 3, or a compact orientable surface. Fix a G-principal bundle P on M and let $\mathcal{A}(P)$ be the* *space of connections on P. The natural pairing*

$$\mu : \mathcal{A}(P) \longrightarrow \mathrm{Hom}\big(g, \Omega^2/B^2\big)$$

given by

$$\mu(A)(f) = F(A) \wedge f + B^2(M)$$

where $F : \mathcal{A}_G(P) \to \Omega^2(M, \mathrm{ad}P)$ is the curvature, is a moment map for the action of the gauge group $\mathcal{G}(P)$ on $\mathcal{A}(P)$ with respect to the symplectic structure $\omega \in \Omega^2(M)$, defined by

$$\omega(\alpha, \beta) = \alpha \wedge \beta + B^2(M)$$

for $\alpha, \beta \in \Omega^1(M, \Omega^2/B^2)$. The reduced space $\mathcal{A}(P)_0$ is the moduli space of flat connection $\mathcal{M}(P) = F^{-1}(0)/\mathcal{G}$ on P, and the reduced form ω_0 takes values in the finite-dimensional vector space $H^2(M)$.

We obtain a similar result in Section 9.4 by incorporating into our analysis the linear characteristic forms on P.

Theorem. *9.4 Let M be a manifold with $\dim M \geq 2$, G a Lie group with $\dim G \geq 2$, P a G-principal bundle on M, and suppose that $\phi \in \mathfrak{g}^*$ is nonzero and Ad^*-invariant. The assignment*

$$(\omega_\phi)_A(\alpha, \beta) = \phi\alpha \wedge \phi\beta + B^2(M), \qquad A \in \mathcal{A}(P),\ \alpha, \beta \in \Omega^1(M, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$$

defines a closed 2-form

$$\omega_\phi \in \Omega^2\big(\mathcal{A}(P),\, \Omega^2(M)/B^2(M)\big)$$

on the space $\mathcal{A}(P)$ of connections on P. The kernel of ω_ϕ at $A \in T_A\mathcal{A}(P)$ is

$$\ker\,(\omega_\phi)_A = \ker\big[\phi : \Omega^1(M, \mathrm{ad}P) \to \Omega^1(M)\big]$$

Moreover, the action of the gauge group $\mathcal{G}$ on $(\mathcal{A}, \omega_\phi)$ is Hamiltonian, with moment map

$$\mu_\phi : \mathcal{A}(P) \to \Omega^2(M)/B^2(M)$$

given by

$$\mu_\phi(A)Y = \phi F(A) \wedge \phi Y + B^2(M), \qquad Y \in \Omega^0(M, \mathrm{ad}P)$$

and the reduced space is

$$\mathcal{A}(P)_0 = (\phi F)^{-1}(0)/\mathcal{G}$$

As a consequence, we obtain a result on the space of complex-traceless connections on a holomorphic vector bundle E over a complex manifold M.

Corollary. *(9.3) Let M be a complex manifold and let E be a holomorphic vector bundle over M with $c_1(E) = 0$. The moduli space of Ricci flat connections is the symplectic reduction of the space of connections $\mathcal{A}(E)$ equipped with the symplectic* *form* ω_{tr} *and moment map given by* $A \mapsto \operatorname{tr} F_A$*.*

In Chapter 10 we turn our attention to the computation of the symplectic volume of the moduli space $\mathcal{M}_G(M)$ of flat G-connections on a symplectic manifold (M, η). Here, we consider the well-known classical symplectic structure on the space of connections $\mathcal{A}(P)$ of a G-principal bundle P over M, given by

$$\omega_A(\alpha, \beta) = \frac{1}{(n-1)!} \int_M \alpha \wedge \beta \wedge \eta^{n-1}$$

for $A \in \mathcal{A}(P)$ and $\alpha, \beta \in \Omega^1(M, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$. It is already known that the $\mathcal{M}(P)$ inherits a symplectic structure when (M, η) is *Lefschetz*, that is, when the map

$$H^1(M, \mathbb{R}) \to H^{2n-1}(M, \mathbb{R})$$
$$\alpha \mapsto \alpha \wedge [\omega]^{n-1}$$

is an isomorphism. It is with respect to thisinduced symplectic structure $\mathcal{M}_G(M)$ that we compute the volume. We also note that a Riemannian structure on M induces a Riemannian structure on $\mathcal{M}_G(M)$, and that our arguments remain valid for the metric-induced volume of $\mathcal{M}_G(M)$.

We first address the case in which the structure group G is abelian. The proof of this result serves as a model for the more technical second case we consider.

Theorem. *(10.1) Let* M *be a symplectic (resp. Riemannian) manifold and let* T *be* *a compact abelian Lie group equipped with an* Ad*-invariant metric. Then*

$$\operatorname{vol}\mathcal{M}_T(M) = \operatorname{vol}(T)^{b_1(M)} \operatorname{vol} H^1(M,\mathbb{Z})\, |\operatorname{Hom}(H_1(M,\mathbb{Z})_{Tor}, T)|$$

where $\operatorname{vol} H^1(M,\mathbb{Z})$ *denotes the lattice covolume of* $H^1(M,\mathbb{Z}) \leq H^1(M,\mathbb{R})$ *with respect to the symplectic (resp. Riemannian) structure on* M *and* $\operatorname{Ch}(H_1(M,\mathbb{Z})_{Tor})$ *is the finite set of characters of* $H_1(M,\mathbb{Z})_{Tor}$.

In Section 10.4 we compute the volume of $\mathcal{M}_G(M)$ when the structure group G is semisimple and the fundamental group $\pi_1(M)$ is free abelian.

Theorem. *(10.2) Let* M *be a symplectic (resp. Riemannian) manifold with free abelian fundamental group* $\pi_1(M)$*,* G *a compact connected semisimple Lie group of dimension* k *and rank* ℓ*,* $\langle\,,\rangle$ *an* Ad*-invariant metric on the Lie algebra* $\mathfrak{g}$*,* H *a maximal torus of* G *with Lie algebra* $\mathfrak{h}$*,* W *the Weyl group,* $\{\alpha\} \subseteq H^*$ *the root system, and* $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$ *the half sum of a subsystem of positive roots. Then*

$$\operatorname{vol}\mathcal{M}_G(M) = \Big(\frac{\operatorname{vol} G}{\sqrt{2\pi}^{\,k-\ell}} \prod_{\alpha>0} \alpha\rho\Big)^{b_1(M)} \frac{1}{|W|} \operatorname{vol} H^1(M,\mathbb{Z})$$

where $\operatorname{vol} H^1(M,\mathbb{Z})$ *denotes the covolume the lattice* $H^1(M,\mathbb{Z})$ *in* $H^1(M,\mathbb{R})$.

We list in Figure 1.1.2 the volumes of the moduli space $\mathcal{M}_G(M)$ for compact semisimple groups G.

In Chapter 11 we study the relation between the moduli spaces $\mathcal{M}_G(M)$ and

G	$\operatorname{vol}\mathcal{M}_G(M)$	
$\mathrm{SU}(r+1)$	$\frac{1}{(r+1)!}\left[\pi^r(2r+2)^{r(r+2)/2}(r+1)^{3/2}\right]^{b_1(M)}$	$\operatorname{vol}H^1(M,\mathbb{Z})$
$\mathrm{SO}(2r+1)$	$\frac{1}{2^r r!}\left[4\pi^r(4r-2)^{r(2r+1)/2}\right]^{b_1}$	$\vdots$
$\mathrm{Sp}(2r)$	$\frac{1}{2^r r!}\left[2\pi^r(4r+4)^{r(2r+1)/2}\right]^{b_1}$	
$\mathrm{SO}(2r)$	$\frac{1}{2^{r-1}r!}\left[\pi r^2(4r-4)^{r(2r-1)/2}\right]^{b_1}$	
E_6	$\frac{1}{3\cdot 4!\,6!}\left[3^{3/2}\pi^6\,24^{39}\right]^{b_1}$	
E_7	$\frac{1}{4!\,4!\,7!}\left[2^{3/2}\pi^7\,6^{133}\right]^{b_1}$	
E_8	$\frac{1}{4!\,6!\,8!}\left[\pi^8\,60^{124}\right]^{b_1}$	
F_4	$\frac{1}{2!\,4!\,4!}\left[2\pi^4\,18^{26}\right]^{b_1}$	
G_2	$\frac{1}{12}\left[3^{1/2}\pi^2\,24^{12}\right]^{b_1}$	

Figure 1.2: The volume of the moduli space $\mathcal{M}_G(M)$ for free abelian $\pi_1(M)$ under the conventions of [6, Ch. VII §13].

$\mathcal{M}_G(\Sigma)$ when Σ is an embedded surface in M. Our main result is that, for a suitable choice of $\Sigma \subseteq M$, there is a symplectic immersion of $\mathcal{M}_G(M)$ into $\mathcal{M}_G(\Sigma)$.

Theorem. *11.2 If $n \geq 2$, then there is a compact, connected embedded surface $\Sigma \subseteq M$ such that $[\Sigma] \in H^2(M)$ is the Poincarè dual of η. The inclusion $i : \Sigma \hookrightarrow M$ yields a symplectic immersion $i^* : \mathcal{M}_G(M) \to \mathcal{M}_G(\Sigma)$. At a connection A on M, the codimension of the image is equal to*

$$\dim\ker\big(\,H^2_A(M,\Sigma;\operatorname{ad}\mathfrak{g}) \longrightarrow H^2_A(M,\operatorname{ad}\mathfrak{g})\,\big)$$

1.2 Background and Motivation

The moduli space of flat connections found its inception in early twentieth-century physics, when it was realized that certain physical fields F could be profitably studied by the introduction of unphysical degrees of freedom corresponding to intrinsic pointwise symmetries of F. Mathematically, this corresponds to the extension of an underlying manifold M to a G-principal bundle P over M.

The aim of this section is to illustrate some of the interplay between geometry and physics, insofar as it relates to the spaces $\mathcal{M}_G(M)$. We begin with the notion of a gauge field theory and conclude with the significance of the symplectic volume of $\mathcal{M}_G(M)$ and an overview of some previous work on its computation.

1.2.1 Gauge Theory and Low Dimensional Geometry

Very generally, a classical *gauge field theory* is a mathematical model of a dynamical process occurring over space-time, which is invariant under the action of a Lie group $\mathcal{G}$ of *gauge symmetries*. This dynamical process frequently takes the form of a differential equation, determined by an action functional, on the space of connections on a G-principal bundle P over a manifold M. The base M represents space-time and the connections are the *gauge fields*. The gauge group $\mathcal{G}$ is the automorphism group $\mathrm{Aut}(P) = P \times_c \mathrm{Aut}(G)$ where $c : G \to \mathrm{Aut}(G)$ is the action of conjugation.

An elegant first example is *Yang-Mills theory*, where the action YM is given by the L^2-norm of the curvature,

$$\mathrm{YM}(A) = \int_M \|F_A\|^2 \,\mathrm{d}vol$$

The solutions of the associated Euler-Lagrange equations, that is, the stationary points of the Yang-Mills functional, are called the *Yang-Mills connections*. Note that the space $\mathcal{M}_G(M)$ is a subspace of the moduli space of G-Yang Mills connections over M.

In a landmark paper [2], Atiyah and Bott noted that YM is an $\mathrm{Ad}\, G$-equivariant Morse function on the space of all connections on a fixed G-principal bundle P. They also noted that $\mathcal{M}_G(\Sigma)$ can be given as the symplectic reduction of $\mathcal{A}_G(\Sigma)$, equipped with a natural symplectic form.

Gauge theory is also relevant to pure mathematics. For example, in his work on *instatons*, Donaldson [15,18] employed an $SU(2)$-gauge theory to obtain smooth non-topological invariants with which he proved the existence of topological 4-manifolds possessing non-diffeomorphic smooth structures. In fact, by his own account, the "application of gauge theory to 4-manifold topology" is one of only two approaches he takes to nearly all his work [17]

Many of Donaldson's proofs were later simplified with the advent of *Seiberg-Witten* theory. Witten [101] recognized the potential of his work in theoretical physics with Seiberg [77] to address general questions in 4-manifold topology. The resulting theory quickly yielded deep results [68]. In his original paper [101] he showed that a broad class of smooth 4-manifolds do not admit metrics of positive scalar curvature. Kronheimer and Mrowka [**?**] applied this theory to prove the Thom conjecture, which stated that a holomorphically embedded complex curve $\Sigma \subseteq \mathbb{C}P^2$ attains the minimum genus over all embedded curves in its homology class. As an application to the study of pseudoholomorphic curves, Taubes [86] used Seiberg-Witten theory to prove the existence of embedded symplectic representatives of certain homology classes on 4-manifolds.

1.2.2 Topological Quantum Field Theory

It can happen that the meaningful information of a gauge theory on M is entirely determined by the homeomorphism type of M. One such case is described here, though it should be noted that this section is largely descriptive.

Definition 1.1. (see [3]) Let Bord_n denote the category of oriented n-dimensional manifolds-with-boundary and cobordisms, and let Λ-Mod be the category of Λ-modules for a unital commutative ring Λ. A *topological quantum field theory* (TQFT) is a functor

$$Z : \mathrm{Bord}_n \to \Lambda\text{-Mod}$$

which is *multiplicative*, that is

$$Z(M_1 \cup M_2) = Z(M_1) \otimes Z(M_2), \text{for objects } M_i$$

$$Z(N_1 \circ N_2) = \big\langle Z(N_1), Z(N_2)\big\rangle, \text{ for morphisms } N_i$$

and *involutory*, that is

$$Z(M^*) = Z(M)^*$$

where M^* is the underlying space of M with the opposite orientation. Note that various further conditions are applied throughout the literature.

As a first example, *Chern-Simons theory* is a TQFT which considers 3-manifolds. Witten [98] established a connection between Chern-Simons theory on the 3-sphere and the Jones polynomial, thus drawing a connection between gauge theory and knot invariants.

When $\omega_{\mathcal{M}_G(M)}$ is integral, general considerations show that there is a complex line bundle with connection $(\mathcal{L}, \nabla)$ over $\mathcal{M}_G(M)$ such that $\omega_{\mathcal{M}_G(M)}$ is equal to the curvature F_∇. The bundle Λ is called a *prequantum bundle* associated to $\omega_{\mathcal{M}_G(M)}$. The holomorphic sections of $\mathcal{L}^{\otimes k}$ are known as the G-theta functions of level k. More generally, for G nonabelian, $\omega_{\mathcal{M}_G(M)}$ is integral. In the context of surfaces Σ, the assignment

$$Z(\Sigma) = H^0(\mathcal{M}_G(\Sigma), \mathcal{L}^k)$$

is essentially the object map of a TQFT for each $k \geq 1$. In addition to their role in TQFT, the spaces $H^0(\mathcal{M}_G(\Sigma), \mathcal{L}^k)$ also appear in *conformal field theory* (CFT) as the *conformal blocks* [76]. The volume of $\mathcal{M}_G(M)$ thus encodes the rate of growth of certain quantities arising from physics.

An interesting "real world" application of TQFTs is the theory of *topological quantum computation*. The underlying idea is to consider the morphisms $Z(M) : Z(\Sigma_1) \to Z(\Sigma_2)$ as information processes [92]. Recent developments in the study of topological phases of matter, resulting in the 2016 Nobel Prize in physics, gives positive evidence for the realization of an actual topological quantum computer in the physical world.

1.2.3 The Volume of the Moduli Space

When $\omega_{\mathcal{M}_G(M)} \in \Omega^2(\mathcal{M}_G(M), \mathbb{R})$ is integral, and under suitable conditions on the space $\mathcal{M}_G(M)$, an application of the Riemann-Roch theorem yields,

$$\operatorname{vol} \mathcal{M}_G(M) = \lim_{k \to \infty} k^{-n} \dim H^0(\mathcal{M}_G(M), \mathcal{L}^k)$$

More generally, we may replace $\mathcal{M}_G(M)$ with any union of suitably regular components. This establishes a connection between geometry, number theory, and quantum field theory.

Closely related to these notions is the *Verlinde formula* for surfaces Σ, which computes the dimensions of the spaces $H^0(\Sigma, \mathcal{L}^k)$ for compact orientable surfaces Σ with certain additional data. Hence, the volume of the moduli space describes the asymptotics of the Verlinde formula.

We conclude this section with a few notes on the development of the computation of the symplectic volume of $\mathcal{M}_G(M)$.

In the language of an SU(2) conformal field theory, the *Wess-Zumino-Witten model*, Thaddeus [88] derived an expression for the volume of certain subspaces of

$\mathcal{M}_{\mathrm{SU}(2)}(\Sigma)$ corresponding to appropriate choices of principal bundles. He also applied the Verlinde formula to nearly determine the rational cohomology of these spaces. Donaldson [16] applied the theory of universal bundles to obtain a purely topological proof with the same results.

Taking a combinatorial approach, Witten [99] used the Reidemeister torsion of an oriented surface Σ to define a volume form on $\mathcal{M}_G(\Sigma)$, with which he computed the volume of components of $\mathcal{M}_G(\Sigma)$ for compact connected semisimple G. Employing his notion of a *symplectic complex*, he established the equality of his combinatorial volume form and the standard symplectic volume. He also determined the combinatorial volume of $\mathcal{M}_G(\Sigma)$ for nonorientable surfaces Σ.

Using his approach to nonabelian localization [51], Liu [52] applied the theory of heat flow on a compact connected semisimple Lie group G to compute the volume of $\mathcal{M}_G(\Sigma)$, for a oriented surface Σ with fixed holonomy of $[A] \in \mathcal{M}_G(\Sigma)$ around the nonempty connected boundary $\partial\Sigma$. Using the same methods, he also determined the intersection numbers of $\mathcal{M}_G(\Sigma)$, thus obtaining the required data for the Verlinde formula.

Part I

Background

These first few chapters are designed to provide motivation, supporting material, and context for Parts II and III. The reader who is so inclined may proceed directly to Part II and refer back to Chapters 2 and 3 as needed for results on principal bundles and symplectic geometry, respectively. Chapter 4 presents a well-known construction that serves as a model for the later development of this work. In anticipation of the symplectic generalization of Part II, Chapter 5 touches briefly on a few other symplectic variants that have already appeared in the literature.

As it is not possible toprovide a review of every subject to be encountered, we give below an outline of suitable references. The portions of this text that require each topic are indicated parenthetically.

- All of the required background on *Lie groups* and *Lie algebras* (every chapter), and in particular the theory of *compact semisimple Lie groups* (Ch. 7 and Part 3), can be found in [7] and [27].

- *Characteristic classes* (Ch. 9) are treated in [43].

- For general background on *complex geometry* (Part 3), see [96] and [69]; for more specialized results (Ch. 9) refer to [43].

- *Local systems* (Chs. 10 and 11) are addressed in [29] and [5].

We close this introduction by way of a warning. Throughout this work, we regularly cite Kobayashi and Nomizu's excellent two-volume reference, *Foundations of Differential Geometry* [42, 43]. However, the formulas that we quote here do not appear as they are written in the original text. The discrepancy arises from the difference between the traditional definition of the wedge product $\wedge : \Omega^k(M) \otimes \Omega^\ell(M) \to \Omega^{k+\ell}(M)$ found in [42, 43], which includes the normalizing factor $\frac{1}{(k+\ell)!}$, and the modern definition, which does not. In this text, we follow the latter convention. Lee [49] notes that the normalized convention is more prevalent in complex geometry.

Chapter 2

Review of Differential Geometry

This chapter provides a reference for certain geometric prerequisites that we shall need later on in the text. The reader in search of a more basic introduction to the field might beneficially consult [49]. For principal bundles, connections, and curvature, we recommend [42]. Banach and Hilbert manifolds are treated in the brief exposition [47].

Throughout this chapter, all manifolds and vector spaces are assumed to be real.

2.1 Banach and Hilbert Manifolds

As the conditions for smoothness will be apparent in the given contexts, we will not have need to invoke the language of Banach and Hilbert manifolds beyond this section. Nonetheless, it is interesting to consider explicitly the theory that identifies, for example, spaces of connections, as infinite dimensional manifolds. As we will show, such manifolds are modeled locally on vector spaces endowed with suitable additional structure.

Definition 2.1. A *topological vector space* is a locally convex, Hausdorff vector space $(U, +, \cdot)$ equipped with a topological structure with respect to which the operations

$$+ : U \times U \to U$$

and

$$\cdot : \mathbb{R} \times U \to U$$

are continuous.

Definition 2.2. A *Banach Space* $(U, \|\cdot\|)$ is a a topological vector space U that is equipped with a norm $\|\cdot\| : U \to \mathbb{R}$ which is both continuous and complete. If $\|\cdot\|$ is induced by an inner product $\langle\, , \rangle : U \otimes U \to \mathbb{R}$, then the pair $\big(U, \langle\, , \rangle\big)$ is called a *Hilbert space*

Definition 2.3. Let $(U_i, \|\cdot\|_i)$ $(i = 0, 1)$ be Banach spaces and consider the function $f : U_0 \to U_1$. When it exists, the *derivative* of f is the map $f' : U_0 \to \mathrm{Hom}(U_0, U_1)$ such that

$$\lim_{h \to 0} \frac{\| f(u+h) - f(u) - f'(u)h \|_1}{\|h\|_0} \;=\; 0$$

for every $u \in U_0$. For $k \geq 1$, we iteratively define the *kth derivative* $f^{(k)}$: $U_0 \to \mathrm{Hom}(U_0^k, U_1)$ of f to be the derivative of $f^{(k-1)}$. Here we identify the spaces $\mathrm{Hom}\big(U_0, \mathrm{Hom}(U_0, U_1)\big)$ and $\mathrm{Hom}(U_0^2, U_1)$ as U_0, U_1 are Banach spaces. The function

f is said to be *smooth* if $f^{(k)}$ exists for all $k \geq 1$.

It is also possible to define differentiation, and hence smoothness, when U is only a topological vector space. This notion, however, is too weak for our purposes.

Definition 2.4. A *smooth manifold* M modeled on the Banach manifold $(V, +, \cdot)$ is a topological space equipped with a collection $(O_i, \phi_i)_{i \in I}$, called an *atlas*, such that

(i) $\{O_i\}_i$ is an open cover of M,

(ii) $\phi_i : O_i \to V$ is a homeomorphism onto its image for each $i \in I$,

(iii) $\phi_j \phi_i^{-1} : \phi(O_i \cap O_j) \to \phi_j(O_i \cap O_j)$ is smooth for each $i, j \in I$ with $O_i \cap O_j \neq \varnothing$.

We define manifolds modeled on Hilbert spaces similarly. The equivalence of all norms on a finite-dimensional vector space U shows that the class of finite-dimensional Banach manifolds is identical to that of classical manifolds modeled on $\mathbb{R}^n$. The role of the Banach and Hilbert theories is thus to function as an extension of the classical theory of manifolds to the infinite-dimensional context. In this regard the extended theory is rather successful. In fact, in the comprehensive introduction to classical differential geometry [48] set entirely in the context of Banach manifolds, it is not until the tenth chapter, on volume forms, that the text specializes to finite dimensions.

Key examples of Banach manifolds include functions spaces and spaces of maps and sections of bundles. The motivating example for our consideration is the space of connections on a principal bundle over a compact manifold, possibly with boundary.

2.2 Fiber and Principal Bundles

Definition 2.5. Let M and F be manifolds and let $\mathrm{Aut}F$ be any Lie subgroup of $\mathrm{Diff}F$. A *fiber bundle* modeled on $(F, \mathrm{Aut}F)$ over M is a smooth map $\pi : E \to M$ such that

(i) E is a smooth manifold,

(ii) every point $x \in M$ has a neighborhood $O \subseteq M$ for which

$$\pi^{-1}O \cong O \times F$$

Any such local diffeomorphism is called a *local trivialization*.

(iii) Given local trivializations over $O, O' \subseteq M$ with $O \cap O'$ nonempty, the transition function

$$\phi : O \cap O' \to \mathrm{Diff}(F)$$

takes its values in $\mathrm{Aut}F$.

We call $\mathrm{Aut}F$ the *structure group* of the fiber bundle E.

For the benefit of the reader, we note that it is not the case that every definition of a fiber bundle utilizes the structure group $\mathrm{Aut}F$. Our construction might be reasonably termed a *structured fiber bundle*; however, we will adhere to the present terminology. We typically relax notation and refer to a fiber bundle E modeled on F, and we occasionally call E an F-fiber bundle.

Definition 2.6. Define the *automorphism bundle* $\mathrm{Aut}E$ of the F-fiber bundle E to be the group of fiberwise automorphisms of E. That is,

$$\mathrm{Aut}_x E = \mathrm{Aut}(E_x)$$

This bundle is occasionally called the *Adjoint bundle* of E and denoted $\mathrm{Ad}E$, about which we will have more to say below. The *gauge group* of E is defined to be the group $\mathcal{G}E$ of sections of $\mathrm{Aut}E$.

The natural action of $\mathcal{G}E$ on the space ΓE of sections of E will be important later on.

Definition 2.7. Let G be a Lie group. A *G-principal bundle* P is a G-topological fiber equipped with a free right action of G, the orbits of which coincide with the fibers of P.

Note in the definition above that a G-topological fiber bundle is fiberwise equivalent to G in the category of manifolds. This does not endow the fibers of P with a group structure.

Definition 2.8. Let P be a G-principal bundle and let F be a topological space equipped with an action $\lambda : G \curvearrowright F$. Define the *associated bundle* $P \times_\lambda F$ to P with *typical fiber* F to be the bundle

$$P \times_\lambda F = (P \times F)/G$$

where G acts diagonally on $P \times F$ by

$$g \cdot (u, f) = (ug^{-1}, gf)$$

Lemma 2.1. *1. Let X be a right G-principal homogeneous space. Then $\operatorname{Aut} X \cong G$ and the isomorphism is canonical precisely if and only if G is abelian.*

2. Suppose that the action $\lambda : G \to \operatorname{Aut} G$ is effective, and that λ commutes with the right regular representation $r : G^{\mathrm{op}} \to \operatorname{Aut} G$. Then λ is equivalent to the left regular representation $\ell : G \to \operatorname{Aut} G$.

3. The left regular action $\ell : G \to \operatorname{Aut} G$ is a group isomorphism.

Proof. 1. Fix $x \in X$. We will show that $\phi : \operatorname{Aut} G \to G$, given by

$$\alpha(x) = x \cdot \phi(\alpha)$$

is a group homomorphism. The map ϕ is well-defined since the right action of G on X is free and transitive. Moreover, ϕ is a homomorphism since

$$\begin{aligned} x \cdot \phi(\alpha\beta) &= \alpha\beta(x) \\ &= \alpha\big[x \cdot \phi(\beta)\big] \\ &= \alpha(x) \cdot \phi(\beta) \\ &= x \cdot \phi(\alpha)\phi(\beta) \end{aligned}$$

Injectivity follows as $\alpha(x) = x \cdot 1_G$ implies $\alpha(x \cdot g) = x \cdot g$ for every $g \in G$, whence $\alpha = 1_{\mathrm{Aut}(X)}$. Finally, since the assignment $\alpha_g : x \cdot h \mapsto x \cdot gh$ determines an automorphism of X, and since $\phi(\alpha_g) = g$, we deduce that ϕ is surjective.

2. We will show that the map $\phi : G \to G$ given by

$$\phi(g) = \lambda_g(1)$$

intertwines λ and ℓ. First note that ϕ is a homomorphism, since

$$\lambda_{gh}(1) = \lambda_g\big[\lambda_h(1)\big] = \lambda_g(1)\,\lambda_h(1)$$

using in turn the facts that λ is a homomorphism and that λ commutes with right multiplication. As

$$\lambda_g\big[\phi(h)\big] = \lambda_{gh}(1) = \phi\big[\ell_g(h)\big]$$

we deduce that ϕ is a morphism from λ to ℓ. Since λ is effective, it follows that ϕ is injective. We conclude that ϕ is an automorphism of G and, consequently, an intertwiner from λ to ℓ.

3. By part 1. there is an isomorphism $\phi : G \xrightarrow{\sim} \mathrm{Aut}\, G$. Under this identification, the usual action of $\mathrm{Aut}(G)$ on G is also given by ϕ. By part 2. ϕ is equivalent

to the left regular action ℓ, yielding an isomorphism $\psi : \operatorname{Aut} G \longrightarrow \operatorname{Aut} G$ such that the following diagram commutes,

$$\begin{array}{ccc} G & \xrightarrow{\ \ell\ } & \operatorname{Aut} G \\ \big\downarrow{\scriptstyle\phi} & & \big\downarrow{\scriptstyle\psi} \\ \operatorname{Aut} G & \xrightarrow{\ 1\ } & \operatorname{Aut} G \end{array}$$

Since ϕ and ψ are isomorphisms, we conclude that that ℓ is an isomorphism as well.

□

We deduce from Lemma 2.1 that the structure group of a G-principal bundle is G.

Proposition 2.1. *Let P be a G-principal bundle. The automorphism bundle* $\operatorname{Aut} P$ *is naturally isomorphic to $P \times_c G$, where $c : G \to \operatorname{Aut} G$ is the action of conjugation.*

Proof. Let X be a right G-principal space and observe that the assignment $x \mapsto [x, 1]_\ell$ determines a canonical isomorphism $i : X \to X \times_\ell G$ of right G-principal spaces from X to $X \times_\ell G$. Since $X \times_c G$ acts naturally on $X \times_\ell G$ by

$$[x, g]_c \cdot [x, h]_\ell = [x, gh]_\ell$$

there is a natural homomorphism $m : X \times_c G \to \operatorname{Aut}(X \times_\ell G)$. For fixed $x \in X$, the maps $\phi : g \mapsto [x, g]_c$ and $\psi : g \mapsto [x, g]_\ell$ yield a commutative square

$$\begin{array}{ccccc} G & \xrightarrow{\quad \ell \quad} & \operatorname{Aut} G & & \\ \big\downarrow{\scriptstyle\phi} & & \big\downarrow{\scriptstyle\psi} & & \\ X \times_c G & \xrightarrow{\ m\ } & \operatorname{Aut}(X \times_\ell G) & \xrightarrow{\ i^*\ } & \operatorname{Aut} X \end{array}$$

where $i^* = \mathrm{Aut}(i^{-1})$. Since ϕ, ψ and ℓ are isomorphisms and since m and i are natural, we deduce that

$$i^* \circ m : X \times_c G \xrightarrow{\sim} \mathrm{Aut}\, X$$

is a natural isomorphism. By applying this fact to each fiber of P, we obtain a family of isomorphisms

$$\chi_x : P_x \times_c G \xrightarrow{\sim} \mathrm{Aut}_x P$$

which is easily seen to be smoothly varying and thus to yield a canonical identification

$$\chi : P \times_c G \xrightarrow{\sim} \mathrm{Aut} P$$

$\square$

Henceforth, we shall identify the bundles $P \times_c G$ and $\mathrm{Aut}P$. In the literature, this bundle is sometimes called the *Adjoint bundle* of P and denoted by $\mathrm{Ad}P$ [42]. The Adjoint bundle $\mathrm{Ad}P$ is equipped with a natural group structure; its Lie algebra $\mathrm{ad}P = P \times_{\mathrm{Ad}} \mathfrak{g}$ is called the *adjoint bundle*. Note that the initial letter of the former bundle is capitalized, while that of the latter is not.

Example 2.1. Let $\pi : E \to M$ be a fiber bundle modeled on F. The *frame bundle* PE of E is the $\mathrm{Aut}F$-principal bundle of fiberwise identifications of F with E. That is,

$$P_x E = \left\{ u : F \xrightarrow{\sim} E_x \right\}$$

The right action of $\mathrm{Aut}F$ on PE is given by

$$(u \cdot g)(f) = u(gf)$$

for $u \in PE$, $g \in \mathrm{Aut}F$, and $f \in F$.

Example 2.2. If E is a V-vector bundle, then the frame bundle PE is isomorphic as an $\mathrm{Aut}V$-principal bundle to the bundle FE of fiberwise bases of E,

$$F_x E = \{(\sigma_i)_i \subseteq E_x \mid (\sigma_i)_i \text{ is a basis of } E_x\}$$

If $V = \mathbb{R}^k$ then this isomorphism is natural and we identify PE and FE.

The following example shows that the class of frame bundles and the class principal bundles coincide.

Example 2.3. If P' is a G-principal bundle on a manifold M, then the frame bundle PP' is canonically isomorphic to P'. To see this, observe that for any $x \in M$ the fiber $P_x P'$ consists of all isomorphisms $\phi_x : G \xrightarrow{\sim} P'_x$ of G-principal homogeneous spaces. Since this collection is itself a principal homogeneous space under the action of the structure group $G \cong \mathrm{Aut}\, G$, we deduce that P' and PP' have equivalent fibers. As

ϕ_x is G-equivariant, it is determined by its evaluation at $1 \in G$ and the global map

$$\varepsilon_1 : PP' \longrightarrow P'$$
$$\phi \longmapsto \phi(1)$$

yields an isomorphism of G-principal bundles.

2.3 Connections and Related Structures

Definition 2.9. Let E be an fiber bundle over a manifold M. The *vertical tangent distribution* $VE \leq TE$ consists of those vectors $X \in TE$ tangent fibers of E. More precisely,

$$V_uE = T_u(E_{\pi u}) \leq T_uE, \qquad u \in E$$

Alternatively, VE is the kernel distribution for the projection $\pi_* : T(TE) \to TE$. A distribution $A \leq TE$ is said to be *horizontal* if A complements the vertical distribution VP, that is, $TE = VE \oplus A$. If A is both horizontal and structure preserving, in the sense that the lift of any $X \in \mathfrak{X}(M)$ to A preserves $\mathrm{Aut}F$, then A is called a *connection.*

Heuristically, a connection on a fiber bundle E is an identification of adjacent fibers.

Example 2.4. Let G be a Lie group and suppose P is a G-principal bundle on a manifold M. At each point $x \in M$, the fiber $\mathrm{Aut}_x P$ coincides with the image of the right action $\rho : G \to \mathrm{Diff}_x P$. Thus, a horizontal distribution $A \leq TP$ is a connection precisely when A is invariant under the right action of G.

Definition 2.10. Let P be a G-principal bundle. A *connection* 1-*form* $\alpha \in \Omega^1(P, \mathfrak{g})$ on P is $\mathfrak{g}$-valued 1-form such that

(i) for each $u \in P$ and $Y \in \mathfrak{g}$, we have $\alpha_u(\underline{Y}_u) = Y$, where $\underline{Y} \in \mathfrak{X}(P)$ is the action-induced vector field of Y on P.

(ii) $\alpha : TP \to \mathfrak{g}$ intertwines the right action of G on TP and the inverse adjoint action Ad^{-1} on $\mathfrak{g}$.

A differential form $\beta \in \Omega^k(M, P)$ is said to be *horizontal* if $\beta(X_i)_i = 0$ whenever $X_i = 0$ for some $i \leq k$.

Definition 2.11. Let $\pi : E \to M$ be a fiber bundle modeled on F, let $I = [0, 1]$, and let $\mathrm{Iso}\, E$ denote the space of fiber identifications $E_x \xrightarrow{\sim} E_y$, $x, y \in M$. A *system of parallel transport*, or a *parallelism*, on E is a smooth map

$$\tau : C^\infty(I, M) \to \mathrm{Iso}\, E$$

such that

$$\tau_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$$

and

$$\tau_{\gamma\cdot\gamma'} = \tau_\gamma \circ \tau_{\gamma'}$$

where $\gamma \cdot \gamma'$ denotes the concatenation of γ and γ'. That is, τ is a morphism of Lie groupoids. We call τ_γ the *parallel transport map*, or *parallel displacement*, along γ.

Definition 2.12. A *covariant derivative* on an F-fiber bundle E on a manifold M is a map $\nabla : \Gamma E \to \Omega^1(M, TE)$ which is compatible with the structure group $\mathrm{Aut}F$ in the sense that

$$\nabla_X \, \sigma g = \sigma(\mathcal{L}_X g) + (\nabla_X \sigma) g$$

for any sections $X \in \mathfrak{X}(M)$, $\sigma \in \Gamma E$, and $g \in C^\infty(M, \mathrm{Aut}\, F)$.

Remark 2.1. If E is a finite-dimensional vector bundle modeled on $\mathbb{R}^k$ then sections of $\mathrm{Aut}\,\mathbb{R}^k$ consist of matrices $(f_{ij})_{i,j\leq k}$ with $f_{ij} \in C^\infty(M)$. By invoking the natural identification $T_u E \cong E_{\pi u}$, $u \in E$, and the linearity inherent in $\mathrm{Aut}\,\mathbb{R}^k$, we consider a covariant derivative as an assignment

$$\nabla : \Gamma E \to \Omega^1(M, E)$$

subject to the condition that

$$\nabla f\sigma = (Xf) \cdot \sigma + f \, \nabla_X \sigma$$

for any $X \in \mathfrak{X}(M)$, $\sigma \in \Gamma E$, and $f \in C^\infty M$.

Theorem 2.1. *Let E be an F-fiber bundle over M, and put $G = \mathrm{Aut}F$. There is a natural equivalence between the following sets:*

(i) $\Omega^1_{\mathrm{ver}}(PE, \mathfrak{g})_G$*, connection* 1*-forms on* P*,*

(ii) $\mathcal{A}(PE)$*, connections on the frame bundle* PE*,*

(iii) $\mathcal{A}(E)$*, connections on* E*,*

(iv) $PT(E)$*, systems of parallel transport on* E*,*

(v) $\mathrm{Cov}(E)$*, covariant derivatives on* E*,*

Proof. We will sketch the correspondences between these sets; the details are not difficult to supply.

(i) $\cong$ (ii). If $\alpha \in \Omega^1(PE, \mathfrak{g})$ is a connection 1-form, then the kernel distribution $\ker \alpha \leq T(PE)$ is horizontal and G-invariant, and therefore forms a connection on PE.

For the reverse inclusion, observe that the connection $A \leq T(PE)$ induces a family of fiberwise projections onto the vertical bundle VP, the fibers of which are canonically identified with $\mathfrak{g}$ by the action of G.

(ii) $\cong$ (iii). This is a consequence of the natural identification $E \cong PE \times_{\mathrm{Aut}F} F$.

(iii) $\cong$ (iv). Suppose $A \in \mathcal{A}(E)$ is a connection on E and $\gamma : I \to M$ is a path. For each $x \in E_{\gamma(0)}$, let $\tilde{\gamma}_x : I \to E$ be the unique lift of γ to E such that $\tilde{\gamma}(0) = x$

and $\tilde{\gamma}$ is tangent to A. The assignment

$$\begin{aligned}\tau_\gamma : E_{\gamma_0} &\longrightarrow E_{\gamma(1)}\\ x &\longmapsto \tilde{\gamma}_x(1)\end{aligned}$$

defines a system of parallel transport τ on E.

Now suppose τ is a system of parallel transport on E. Then the assignment which sends a point $x \in E$ to the horizontal subspace of T_xE which is tangent to all horizontal curves through x defines a connection on E.

(iv) $\cong$ (v). Suppose τ is a system of parallel transport on E. Let $X \in \mathfrak{X}(M)$ and $\sigma \in \Gamma E$. Choose a path $\phi : I \to \mathrm{Diff}(M)$ so that X is tangent to ϕ at 0, and let $\tau_0^t : E \to E$ denote the parallel transport along the flow of ϕ from 0 to t. The assignment of the derivative of $\sigma - \tau_0^t\sigma$ at 0 to the pair (X, σ) defines a covariant derivative ∇ on E.

Now suppose ∇ is a covariant derivative on E and $\gamma : I \to M$ is a path. Similar to an argument above, for each $x \in E_{\gamma(0)}$ denote by $\tilde{\gamma}_x : I \to E$ the lift of γ to E such that $\nabla_{\dot{\gamma}}\tilde{\gamma} = 0$. Then the assignment $\tau_\gamma : x \mapsto \tilde{\gamma}_x(1)$ defines a system of parallel transport. $\square$

Remark 2.2. It is interesting to note that, appearing in the early twentieth century, the first of these equivalent notions was that of parallel transport on vector bundles, specifically in the presence of a Riemannian structure as motivated by considerations from physics. It was Weyl who first abstracted the idea of parallel transport from the underlying metric structure. See [90] for a brief but thorough history of the development of the theory of connections and, in particular, for references for this remark.

Remark 2.3. In light of Theorem 2.1, it is commonplace to refer to a covariant derivative ∇ on a vector bundle E as a "connection" on E. See, for example, [14]. We will occasionally follow this convention, and we will write $\mathcal{A}(E)$ for the space $\mathrm{Cov}(E)$ of covariant derivatives on E.

Henceforth, we shall only consider covariant derivatives in the context of vector bundles.

Proposition 2.2. *Let M be a smooth manifold, let G be a Lie group, and suppose that P is a G-principal bundle on M. The space of connections $\mathcal{A}(P)$ on P is naturally an $\Omega^1(M, \mathrm{ad}P)$-affine space.*

Proof. Let $\alpha, \alpha' \in \Omega^1(P, \mathfrak{g})$ be connection 1-forms and let $\alpha \in \Omega^1(M, \mathrm{ad}P)$, which we identify with the space of G-equivariant horizontal 1-forms on P. Being the linear combination of G-equivariant 1-forms, it follows that $\alpha + \beta$ and $\alpha - \alpha'$ are G-equivariant. Moreover,

$$(\alpha + \beta)_u(\underline{Y}_u) = Y$$

from which we deduce that the action of $\Omega^1(M, \mathrm{ad}P)$ preserves the space of connection

1-forms, and

$$(\alpha - \alpha')_u(\underline{Y}_u) = 0$$

which implies that $\alpha - \alpha' \in \Omega^1(M, \mathrm{ad}P)$ and, consequently, that the action of $\Omega^1(M, \mathrm{ad}P)$ is transitive. The result follows by the natural equivalence $\mathcal{A}(P) \cong \Omega_{\mathrm{ver}}(P, \mathfrak{g})_G$ of Theorem 2.1. □

2.4 Curvature and Holonomy

Definition 2.13. Let E be a fiber bundle modeled on F over M, let $A \in \mathcal{A}(E)$ be a connection on E, and fix $x \in M$. The *holonomy group* $\mathrm{Hol}_{A,x}$ is defined to be the subgroup of $\mathrm{Aut}E_x \cong \mathrm{Aut}F$ which is generated by the parallel transport of the fiber E_x around all closed curves $\gamma \in C^\infty(I, M)$ based at x, that is, $\gamma(0) = \gamma(1) = x$.

If $G \leq \mathrm{Aut}_x E$ and $\mathrm{Hol}_{A,x} \leq G$, then A is said to be a *G-connection* on E.

We note that the isomorphism $\mathrm{Aut}P_x \cong \mathrm{Aut}F$ is natural only up to the action of conjugation on $\mathrm{Aut}F$. Additionally, by an implicit and noncanonical identification of an arbitrary fiber E_x, $x \in M$, with F, we occasionally refer to G-connections $A \in \mathcal{A}(E)$, where $G \leq \mathrm{Aut}F$.

In light of Theorem 2.1, we also note that we may refer to the holonomy of any of the connection-equivalent structures on E or PE.

Definition 2.14. Let M be a manifold, let G be a Lie group, and let P be a G-principal bundle over M. The *exterior covariant derivative* with respect to the connection $A \in \mathcal{A}(P)$ is the operator

$$\mathrm{d}_A : \Omega^*(P) \to \Omega^{*+1}(P)$$

given by

$$\mathrm{d}\beta(hX_1, \ldots, hX_{k+1})$$

where $\beta \in \Omega^k(P)$ is a differential k-form on P and $h : TP \to A$ is the *horizontal projection* of the fibers of TP to the distribution $A \leq TP$.

The *curvature* $F_A \in \Omega^2(P, \mathfrak{g})$ of A is the exterior covariant derivative $\mathrm{d}_A\alpha$ of the connection 1-form $\alpha \in \Omega^1(P, \mathfrak{g})$ with respect to A. The connection A is said to be *flat* if $F_A = 0$.

Remark 2.4. Some authors, for example, [42], define a flat connection $A \in \mathcal{A}(P)$ to be one which is locally equivalent to the *canonical flat connection* $\pi_2^*TG \leq TP$ on the product G-principal bundle $P = M \times G$. This condition is equivalent to the vanishing of the curvature F_A of A.

Yet another equivalent condition is that the distribution $A \leq TP$ is integrable. Consequently, flat connections are occasionally called *integral connections*.

We also state the familiar definition of curvature of a covariant derivative.

Definition 2.15. Let M be a manifold and let E be a vector bundle on M. The *curvature* $R_\nabla \in \Omega^2(M, \mathrm{End}E)$ of a covariant derivative ∇ on E is defined to be

$$R_\nabla(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$$

Remark 2.5. Of course, it is not immediately clear from the defining expression that R_∇ is indeed a 2-form with values in $\mathrm{End}E$. However, this is well-known and in particular we presume it to be familiar to the reader. If not, the text [14] will be a helpful resource.

We also refrain from establishing the appropriate equivalence between the curvature of the covariant derivative ∇ on E and that of the corresponding connection A on the frame bundle PE. For this, we refer to [42].

Let us collect now a few results which we will have cause to apply later in the text.

Lemma 2.2. *Let P be a G-principal bundle on a manifold M, and let A be a connection on P.*

(i) For any $X, X' \in \mathfrak{X}(P)$ and $g \in G$, we have $F_A(g_*X, g_*X') = \mathrm{Ad}_g\, F_A(X, X')$.

(ii) The curvature $F_A \in \Omega^2(P, \mathfrak{g})$ on P descends to a form $\pi_*F_A \in \Omega^2(M, \mathrm{ad}P)$ on M, where we recall that $\mathrm{ad}P = P \times_{\mathrm{Ad}} \mathfrak{g}$.

(iii) The exterior covariant derivative $\mathrm{d}_A : \Omega^*(P, \mathfrak{g}) \to \Omega^{*+1}(P, \mathfrak{g})$ descends to an operator $\pi_*\mathrm{d}_A : \Omega^*(M, \mathrm{ad}P) \to \Omega^*(M, \mathrm{ad}P)$.

(iv) If the connection A is flat, then $\mathrm{d}_A^2 : \Omega^*(P, \mathfrak{g}) \to \Omega^{*+2}(P, \mathfrak{g})$ *is the zero operator.*

(v) If A if flat, then $(\pi_* \mathrm{d}_A)^2 = 0$ *on* $\Omega^*(M, \mathrm{ad}P)$.

Proof. (i) We adapt the approach of Chapter 2, Section 5, of [42]. Since the action of G preserves the fibers of P, it follows that G preserves the vertical distribution $VP \leq TP$. Thus, if either of X or X' is vertical, then

$$\mathrm{d}\alpha(hg_*X, hg_*X') = 0 = \mathrm{Ad}_g \, \mathrm{d}\alpha(hX, hX')$$

If, on the other hand, both X and X' are horizontal, then so too are g_*X and g_*X'. In this case, each of X, X', g_*X, g_*X' is fixed by h and vanishes under α, so that

$$\mathrm{d}\alpha(hg_*X, hg_*X') = \alpha\big(g_*[X, X']\big) = \mathrm{Ad}_g \, \alpha\big([X, X']\big) = \mathrm{d}\alpha(X, X')$$

In either of these two cases, we have $F_A(g_*X, g_*X') = \mathrm{Ad}_g \, F_A(X, X')$. The result follows as every vector field $X \in \mathfrak{X}(P)$ splits as the sum of its vertical and horizontal components, and since F_A is multilinear.

(ii) This follows from (i) and the definition of $\mathrm{ad}P$.

(iii) This immediately from the identification of $\mathrm{ad}P$ with the space of G-invariant horizontal $\mathfrak{g}$-valued forms on P. For details on this identification, see, for example, Chapter 2, Section 5, of [42].

(iv) Let us first recall Proposition 3.11 of Chapter 1 in [42]: namely, that if N is a manifold, $\eta \in \Omega^k(N)$, and $X_i \in \mathfrak{X}(N)$ for $0 \leq i \leq k$, then

$$\begin{aligned} \mathrm{d}\omega(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i \, X_i \, \omega(X_0, \dots, \widehat{X}_i, \dots, X_k) \\ + \sum_{0 \leq i < j \leq k} \omega\big([X_i, X_j], X_0, \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots, X_k\big) \end{aligned}$$

Note that we omit the factor of $\frac{1}{k+1}$ that appears in [42], and which arises as the normalization factor for the traditional definition of the wedge product on forms, to conform with the currently more popular convention which does not include it.

Now suppose that the connection $A \in \mathcal{A}(P)$ is flat, that $\beta \in \Omega^k(P, \mathfrak{g})$, and that $X_i \in \mathfrak{X}(P)$, $i \leq k$. As $\mathrm{d}_A^2(X_i)_i$ vanishes if any X_i is vertical, suppose that each X_i is horizontal. Invoking an equivalent characterization of flatness, we consider $A \leq TP$ as an integral distribution on TP, and we deduce that $[X_i, X_j] \in A$ for each $i, j \leq k$. Consequently, the horizontal projection map h fixes each $X_i, [X_i, X_j] \in \mathfrak{X}(P)$, and we conclude that the expansion of $\mathrm{d}_A^2\beta(X_i)_i$, by means of two applications of the above formula, is identical to the corresponding expansion of $\mathrm{d}^2\beta(X_i)_i = 0$.

(v) This follows from (iii) and (iv).

□

Remark 2.6. The forms $\pi_* F_A$ and $\pi_* \mathrm{d}_A$ are usually denoted simply by F_A and d_A. In fact, we will usually consider the curvature and exterior covariant derivative as maps

$$F : \mathcal{A}(P) \longrightarrow \Omega^2(M, \mathrm{ad}P)$$

and

$$\mathrm{d}_A : \Omega^*(M, \mathrm{ad}P) \longrightarrow \Omega^{*+1}(M, \mathrm{ad}P)$$

2.5 Flat Connections and Character Varieties

We are now ready to define one of the central constructions of this dissertation.

Definition 2.16. Let M be a manifold, let E be a fiber bundle on M modeled on F, and suppose that $G \leq \mathrm{Aut}F$. Define the *moduli space of flat G-connections* on E to be the quotient $\mathcal{M}_G(E) = \mathcal{A}(E)_{F=0}/\mathcal{G}E$ of the space of flat connections $\mathcal{A}(E)_{F=0}$ by the action of the gauge group $\mathcal{G}E$.

Definition 2.17. Fix a manifold M and a Lie group G. Let $\mathcal{P}_G(M) = \{P_i\}_i$ denote a fixed collection of G-principal bundles on M such that $\mathcal{P}_G(M)$ contains precisely one representative from each isomorphism class of G-principal bundles on M. Let $\mathcal{A}_G(M) = \bigcup_i \mathcal{A}(P_i)$ denote the union of the spaces of connections, and define the *moduli space of flat G-connections on M* to be the union $\mathcal{M}_G(M) = \bigcup_i \mathcal{M}(P_i)$ be the disjoint union of the moduli spaces $\mathcal{M}(P_i)$.

Remark 2.7. We sometimes heuristically characterize $\mathcal{M}_G(M)$ as the moduli space of all flat connections from any G-principal bundle P over M. This characterization is not literally true as, for example, the class of all G-principal bundles over M does not even form a set.

The choice of representatives involved in our definition of $\mathcal{M}_G(M)$ may be taken to be problematic, as there is no natural identification between the space of connections $\mathcal{A}(P)$ and $\mathcal{A}(P')$ for distinct isomorphic G-principal bundles P and P' on M. However, the ambiguity inherent in the construction of $\mathcal{M}_G(M)$ will not be relevant for our purposes, and we refer to $\mathcal{M}_G(M)$ as *the* moduli space of flat G-connections on M.

Fix a connected manifold M and a Lie group G. Our present aim is to establish a natural equivalence between the moduli space $\mathcal{M}_G(M)$ to the *character variety* $\mathrm{Hom}(\pi_1 M, G)/G$, where G acts on $\mathrm{Hom}(\pi_1 M, G)$ by conjugation. More precisely,

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}$$

for $g \in G$, $\rho \in \mathrm{Hom}(\pi_1 M, G)$, and $\gamma \in \pi_1 M$. As M is connected, we will tend to leave implicit the basepoint $x \in M$ for the fundamental group $\pi_1 M$. The particular choice of $x \in M$ will have no bearing on our results.

We will prove this result for a linear group G, in the classical context of vector bundles. The treatment of an arbitrary Lie group, and in the context of principal bundles, is both analogous and notably more straightforward. A partial treatment can be found in Chapter 2, Section 9, of [42].

Theorem 2.2. *Let G be a linear group. There is a well-defined map*

$$\begin{aligned}\mathrm{Hol} : \mathcal{M}_G(M) &\longrightarrow \mathrm{Hom}(\pi_1 M, G)/G \\ \big[(\nabla, E)\big]_{\mathcal{G}} &\longmapsto [\mathrm{Hol}_\nabla]_G\end{aligned}$$

which assigns to each gauge-equivalence class of a G-connection ∇ the holonomy representation $\mathrm{Hol}_\nabla \in \mathrm{Hom}(\pi_1 M, G)$.

Proof. Fix a basepoint $x \in M$ and identify the structure group G as a subgroup of $\mathrm{Aut}E_x$.

First, we will show that Hol_∇ a well-defined map on $\pi_1 M$. Second, we will prove that Hol_∇ depends only on the gauge-equivalence class of ∇.

Fix a basepoint $x \in M$, let $\gamma(t)$ be a loop in M based at x, and let $\gamma_r(t)$ be a smooth variation of $\gamma(t)$. Write ∂_r for the variational vector field along γ and observe that

$$0 = R(\partial_r, \gamma') = \nabla_{\partial_r}\nabla_{\gamma'} - \nabla_{\gamma'}\nabla_{\partial_r}$$

Denoting by $s_r(t) = \tau_{\gamma_r|[0,t]}$ the parallel transport of any $v \in E_x$ along γ_r, we have

$$\nabla_{\gamma'}\nabla_{\partial_r} s = \nabla_{\partial_r}\nabla_{\gamma'} s = 0$$

and so

$$\nabla_{\partial_r} s|_{t=1} = \tau_\gamma(\nabla_{\partial r} s|_{t=0}) = 0$$

Thus $\tau_\gamma \in G$ is constant as a function on $[\gamma] \in \pi_1(M,x)$. We conclude that the holonomy of ∇ is a well-defined homomorphism from $\pi_1(M,x)$ to G.

Again let $\gamma : I \to M$ be a loop based at x, and let $s(t)$ be a parallel section of E over γ. We deduce that for any $u \in \mathcal{G}E$,

$$\nabla^u_{\gamma'} us = u\nabla_{\gamma'} s = 0$$

It follows that $us(1)$ is the parallel transport of $us(0)$ around γ. Since u, s, and γ are arbitrary, we have

$$\tau^u_\gamma s = (\tau_\gamma s)^{u(x)}$$

and hence $\mathrm{Hol}(\nabla^u) = \mathrm{Hol}(\nabla)^{u(x)}$. This completes the proof. $\square$

Theorem 2.3. *Let G be a linear group. The holonomy map* $\mathrm{Hol} : \mathcal{M}_G(M) \to \mathrm{Hom}(\pi_1 M, G)/G$ *establishes a bijection between the moduli space of flat G-connections $\mathcal{M}_G(M)$ and the character variety* $\mathrm{Hom}(\pi_1 M, G)/G$.

Proof. Suppose that $\nabla, \bar{\nabla}$ induce identical holonomy $\mathrm{Hol}_\nabla = \mathrm{Hol}_{\bar{\nabla}}$ and define the section $u \in \Gamma E$ by

$$u(y) = \bar{\tau}_\alpha \tau_\alpha^{-1}$$

where α is any path connecting x to y. To see that $u(y)$ is independent of α, consider a second path β joining x to y, and note that $\mathrm{Hol}_\nabla = \mathrm{Hol}_{\bar\nabla}$ implies that

$$\tau_\beta^{-1}\tau_\alpha = \bar\tau_\beta^{-1}\bar\tau_\alpha$$

which yields

$$\bar\tau_\beta\tau_\beta^{-1} = \bar\tau_\alpha\tau_\alpha^{-1}$$

Using the relation $\bar\tau_\alpha = u\tau_\alpha$ and writing τ (resp. $\bar\tau$) for the parallel transport operators with respect to ∇ (resp. $\bar\nabla$) along α, we deduce that

$$\begin{aligned}
\bar\nabla_{\alpha'}s &= \lim_{h\to 0}\frac{\bar\tau_0^t(\bar\tau_0^{t+h})^{-1}s - s}{t} \\
&= \lim_{h\to 0}\frac{u\tau_0^t(\tau_0^{t+h})^{-1}u^{-1}s - s}{t} \\
&= u\lim_{h\to 0}\frac{\tau_0^t(\tau_0^{t+h})^{-1}u^{-1}s - u^{-1}s}{t} \\
&= u\nabla_{\alpha'}u^{-1}s
\end{aligned}$$

for any section $s \in \Gamma E$. Therefore, the Hol is injective.

Now suppose that $\rho \in \mathrm{Hom}(\pi_1 M, G)$ is a representation. We will construct a flat connection ∇ on E with holonomy ρ^{-1}. Consider the action of $\pi_1 M$ on the trivial vector bundle $\tilde M \times V$ given by

$$\sigma \cdot (\tilde x, v) = \big(\sigma\tilde x, \rho(\sigma)v\big)$$

where $\pi_1 M$ acts on $\tilde{M}$ in the first factor by deck transformations. Since the action of $\pi_1 M$ on $\tilde{M}$ is free and proper, it follow that the action of $\pi_1 M$ on $\tilde{M} \times V$ is free and proper as well and, consequently, that the quotient E is a smooth manifold. Moreover, as the projection $p_1 : \tilde{M} \times V \to \tilde{M}$ is $\pi_1 M$-equivariant, it descends to a projection $\pi : E \to M$. Since $\pi_1 M$ preserves the G-structure on the fibers of $M \times V$, it follows that $\pi : E \to M$ is a G-vector bundle on M.

Fix $s \in \Gamma E$ and $X \in \mathfrak{X}(M)$, and let $\tilde{s} : \tilde{M} \to V$ and $\tilde{X} \in \mathfrak{X}(\tilde{M})$ be the corresponding lifts to $\tilde{M}$. Since

$$\sigma \cdot (\tilde{X}\tilde{s}) = (\sigma \cdot \tilde{X})(\sigma \cdot \tilde{s}) = \tilde{X}\tilde{s}$$

for all $\sigma \in \pi_1 M$, it follows that $\tilde{X}\tilde{s}$ descends to a section of E. As the assignment $(X, s) \mapsto \nabla_X s = \tilde{X}\tilde{s}$ is tensorial in X and linear in s, it follows that ∇ is a connection on E. Moreover, the curvature of ∇ vanishes as $R(X, Y)s$ is represented by the section

$$\tilde{X}\tilde{Y}\tilde{s} - \tilde{Y}\tilde{X}\tilde{s} - \widetilde{[X, Y]}\tilde{s} = 0$$

of the trivial bundle $\tilde{M} \times V$.

Let γ be a loop based at $x \in M$ and let s be a parallel section of E over γ. Hence,

$$\tilde{\gamma}'\tilde{s} = 0$$

where $\tilde{\gamma}$ is any lift of γ. It follows that $\tilde{s} : \tilde{M} \to V$ is a constant function on $\tilde{\gamma}$. Consequently,

$$
\begin{aligned}
s(1) &= \left[\tilde{\gamma}(1), \tilde{s}(1)\,\right] \\
&= \left[\gamma \cdot \tilde{\gamma}(0), \tilde{s}(0)\right] \\
&= \left[\tilde{\gamma}(0), \gamma^{-1} \cdot \tilde{s}(0)\right] \\
&= \gamma^{-1} \cdot s(0)
\end{aligned}
$$

and we conclude that $\mathrm{Hol}_{\nabla} = \rho^{-1}$. Since $\rho \in \mathrm{Hom}(\pi_1 M, G)$ is arbitrary, it follows that Hol is surjective. $\square$

We note that this bijection is in fact a diffeomorphism on the regular parts, when each space is equipped with the natural smooth structure.

Chapter 3

Symplectic Geometry and Reduction

This chapter comprises a whirlwind review of symplectic geometry. We begin with symplectic vector spaces and finish with remarks on the structure theory of symplectic reduced spaces.

Excellent introductions to this material are [8], [65], and [4]. The latter is especially useful as it also presents a thorough treatment of the moduli space of flat connections on a compact oriented surface. Also highly recommended is the expository paper [94], which takes a high-level perspective and provides motivation for and insight into much of the material. For an advanced reference, the monograph [93] is recommended. For mathematically-complete expositions with a view towards classical mechanics, the reader may find the introductory treatise [62] or the more more advanced notes [60] useful.

3.1 Historical Remarks

The setting for symplectic geometry originated in the mid–eighteenth- and early nineteenth-century work of mathematicians on the Kepler problem (see [58] for this and the following paragraph), the aim of which is the determination of the motion of n planets as they orbit around the sun. Contemporary mathematicians took a two-step approach. First, they observed that the space of simplified solutions, that is, the elliptic solutions for the unrealistic model in which it is only the sun that exerts a gravitational pull, locally forms a $6n$-dimensional smooth manifold, which we shall call M. Second, they approximated the original problem as that of the determination of the evolution on M of a simplified solution p under the effects of the mutual gravitational interaction of the planets. Up to this approximation, the Kepler problem was thus reduced to the solution of a differential equation on M.

In a paper of 1808 [46] *Joseph-Louis Lagrange* published a breakthrough discovery: that, when suitably expressed in terms of certain functions of the coordinates, the previously intractable equations of motion become considerably simplified. These functions of the coordinates, now called the *Lagrange parentheses*, formed the components of a natural symplectic structure on M, though we note that Lagrange did not identify it as such. In 1811 he incorporated his work on the Kepler problem in the second addition of his hugely impactful *Méchanique Analytique*[1] [45] in which he purports to reduce all mechanical phenomena to pure numerics. We present the vector-valued counterpart of this mechanical formalism in Chapter 8.

Inspired by the work of Lagrange, *William Rowan Hamilton* sought to extend his methods to the field of optics (see [21]). In this pursuit, he discovered that the solutions to the guiding equations were implicit in a single function, the knowledge of which would suffice to determine the path of a luminous ray. When he later returned to mechanics, this insight undoubtedly facilitated his realization, published in his *First Essay on a General Method in Dynamics* [28] of 1834, that the dynamics of a mechanical system are also encoded in a single function – the *Hamiltonian function* of the system. As it transpired, Hamilton's proofs were soon to be challenged by *Carl Gustav Jacob Jacobi*; and it was Jacobi who was ultimately responsible for the classical form of the theory [21].

Motivated by the success of Galois theory in the study of algebraic equations, *Sophus Lie* undertook to develop a theory of symmetry that would play a similar role for differential equations [30]. His efforts in this regard initiated the study of *Lie groups*. While not primarily interested in mechanics, it has been noted [62, 95] that many of the key results in symplectic geometry were prefigured in the second volume of Lie's *Theorie der Transformationsgruppen*[2] [50] of 1890. This includes the Poisson structure on the dual of a Lie algebra as well as the moment map for an action on

[1] *Analytical Mechanics*
[2] *Theory of Transformation Groups*

$\mathbb{R}^{2n}$.

It was not until *Hermann Weyl* published *The Classical Groups* [97] in 1939 that the term "symplectic" first appeared in mathematics, in reference to the symplectic group (see [93]). Weyl, who regretted the confusion which inevitably followed his original choice of "complex group", settled on "sym-plectic" as the Greek etymological equivalent of "com-plex" [97]. The first component of each corresponds to "together", the second to "plait", that is, a braid. It is noted in [8] that the term "symplectic" had already acquired a meaning in English, prior to Weyl's intervention: it refers to a bone in the head of a fish.

Beginning in the 1960s, *Bertram Kostant* [44], *Jean-Marie Souriau* [83] and Alexandre Kirillov [40] were interested in applying symplectic geometry to problems in quantization and linear representations. This was the first time the moment map was explicitly identified; Souriau called it *l'application moment*[3]. We refer to [62] for further details on this topic.

In his *Topology and Mechanics* [81] of 1970, *Stephen Smale* saw himself in the legacy of Emmy Noether when he defined the *angular momentum* $J : TM \to \mathfrak{g}^*$ of a mechanical system (M, K, V) with symmetries G, and showed it to be preserved by the dynamics. Here, M is a configuration space with Riemannian metric K, and $V : M \to \mathbb{R}$ represents the potential energy of the system. Under the identification $K : TM \xrightarrow{\sim} T^*M$, the assignment J is the canonical moment map for the induced

[3] *the application of momentum*

action of G on the cotangent bundle T^*M with the standard symplectic structure. Smale chose the Kepler problem as his motivation [81, 82], an example which has already proven instrumental in the field. To avoid confusion, we also note that, in this text, the configuration space M will be denoted by Q, and the cotangent bundle T^*M will be denoted by M.

The current forms of the reduction theorems were first obtained in 1974 by *Jerrold Marsden* and *Alan Weinstein* [59], and independently by *Kenneth Meyer* [67], as a synthesis of the previous work of Kostant and Smale. Marsden and Weinstein define a *moment* $\psi : M \to \mathfrak{g}^*$ of a symplectic manifold (P, Ω) under the compatible action of G via the equivalent condition of Proposition 3.2. They credit their definition to Souriau [84] and they show that, under the appropriate smoothness conditions, there is a canonical symplectic structure on the *reduced phase space* $P_\mu = \psi^{-1}(\mu)/G_\mu$. This is precisely the Stabilizer Reduction Theorem below. Their use of the term "moment", which coincides with our notion of a "weak moment map", was a direct transcription of Souriau's "application moment"; the English translation "momentum map" remains a popular alternative today. It has been lightheartedly claimed [63] that a given mathematician's preference between the two is correlated with the North American coast by which they might reside: "moment" on the east coast, "momentum" on the west. In the same paper, Marsden and Weinstein consider the comoment map $\widehat{\psi} : \mathfrak{g} \to C^\infty(P)$, which they do not name but only characterize as the "dual" of the moment $\psi : M \to \mathfrak{g}$, thus accounting for its present nomenclature.

For additional historical details, see [58] for the early stages of the field, and see [62] for subsequent progress. We refer to [21] for a comprehensive treatment of the parallel development of classical mechanics. Chapter 5 of this text outlines a few of the advances following the work of Marsden, Weinstein, and Meyer.

3.2 Symplectic Vector Spaces

Definition 3.1. A *symplectic structure* on a vector space U is an alternating bilinear form $\omega : \Lambda^2 U^* \to \mathbb{R}$ which is *nondegenerate* in the sense that the map

$$\begin{aligned} U &\longrightarrow U^* \\ u &\longmapsto \omega(u, \cdot) \end{aligned}$$

is nondegenerate, that is, injective.

Remark 3.1. There is another notion of nondegeneracy of ω, which is equivalent for finite-dimensional U, but strictly stronger in the infinite dimensional case. Specifically, we could alternatively require that the assignment $u \mapsto \omega(u, \cdot)$ constitute an isomorphism $U \xrightarrow{\sim} U^*$. Let us call this *strong nondegeneracy.* As it is this latter condition that is utilized by the symplectic reduction theorems, we may reasonably conclude this to be the proper notion of nondegeneracy. There are two justifications for our formulation in Definition 3.1. Firstly, our present choice conforms with the conventions of most of the literature; though we note that, in the original paper on symplectic reduction, Marsden and Weinstein refer to our symplectic structure as a *weak symplectic structure*. Secondly, and more importantly for our purposes, it is easily seen that the class of alternating V-valued forms $\omega \in \Lambda^2 U^* \otimes V$ which establish an isomorphism $U \xrightarrow{\sim} U^* \otimes V$ is empty when $\dim V \geq 2$.

Remark 3.2. In fact, strong nondegeneracy is not required, even for infinite-dimensional applications. The interested reader can refer to [59] for the appropriate conditions in the infinite-dimensional context.

Example 3.1. Let W be a vector space and put $U = W \oplus W^*$. Then U possesses a natural symplectic structure ω given by

$$\omega(u + \alpha, u' + \alpha') = \alpha'(u) - \alpha(u')$$

for $u, u' \in U$ and $\alpha, \alpha' \in U^*$.

When $U = \mathbb{R}^n$, this is typically presented as follows. Equip $\mathbb{R}^{2n}$ with linear coordinates $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ and put $\omega = \sum_{i=1}^n \mathrm{d}x^i \wedge \mathrm{d}y^i$. The pair $(\mathbb{R}^{2n}, \omega)$ forms a symplectic vector space. In terms of the coordinate basis corresponding to our choice of coordinates, the symplectic structure ω takes the form

$$\begin{pmatrix} & & & & 1 & & & \\ & & & & & 1 & & \\ & & & & & & \ddots & \\ & & & & & & & 1 \\ -1 & & & & & & & \\ & -1 & & & & & & \\ & & \ddots & & & & & \\ & & & -1 & & & & \end{pmatrix}$$

In fact, up to linear equivalence, this is the *only* symplectic structure on a finite-dimensional vector space.

This is in a certain sense the primary example of a symplectic structure, as it naturally evokes the canonical symplectic form on the cotangent bundle of a manifold modeled on U, which itself formed the original motivation for the field of symplectic geometry. We will return to this key example in the vector-valued context in Chapter 8.

Example 3.2. Let Σ be an oriented closed surface and consider the infinite-dimensional vector space $\Omega^1(\Sigma)$. The wedge product and integration yield a symplectic structure $\omega \in \Lambda^2\Omega^1(\Sigma)^*$,

$$\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta, \qquad \alpha, \beta \in \Omega^1(\Sigma)$$

The pair $\big(\Omega^1(\Sigma), \omega\big)$ is the prototypical example of an infinite-dimensional symplectic vector space.

Definition 3.2 (symplectic orthogonal)**.** Let ω be a symplectic structure on U and let $\bar{U} \leq U$ be a linear subspace. The *symplectic orthogonal* of $\bar{U}$ is the subspace

$$\bar{U}^{\omega} = \{u \in U \,|\, \omega(u, \bar{u}) = 0 \text{ for all } \bar{u} \in \bar{U}\}$$

Lemma 3.1. *If $U \leq V$ and V is finite-dimensional, then $U^{\omega\omega} = U$.*

Proof. First observe that

$$u \in U \implies \forall v \in U^{\omega} : \omega(u, v) = 0 \implies u \in U^{\omega\omega}$$

yields $U \leq U^{\omega\omega}$. The result will follow by establishing the equality of the dimensions of U and $U^{\omega\omega}$.

Note that U is the pullback by $\iota\,\omega : V \to V^*$ of the annihilator $U^0 \leq V^*$. Now if $\{u_i\}_i$ is a basis U and $\{u_i, v_j\}_{i,j}$ is a basis of V, then $\{v_j^*\}_j \subseteq V^*$ is a basis of $U^0 \leq V^*$. Thus $\dim U + \dim U^0 = \dim V$, and so U and U^{ω} have complementary dimension in V. We thus obtain the pair of equalities

$$\dim U + \dim U^{\omega} = \dim V$$
$$\dim U^{\omega\omega} + \dim U^{\omega} = \dim V$$

whence $\dim U = \dim U^{\omega\omega}$, as required. $\square$

Definition 3.3. Let (V, ω) be a symplectic vector space and $U \leq V$ a subspace. We define the following terms by the corresponding conditions,

term	condition
isotropic	$U \leq U^{\omega}$
coisotropic	$U^{\omega} \leq U$
Lagrangian	U is maximal isotropic
symplectic	$(U, \omega\|_U)$ symplectic

Thus, ω is isotropic when ω vanishes on U; coisotropic when ω vanishes on U^{ω}, that is, when U^{ω} is isotropic; Lagrangian when U is minimal coisotropic or, equivalently, when U is both isotropic and coisotropic; and symplectic when ω is nondegenerate on U.

We will assume, for the remainder of this chapter, that all vector spaces are finite-dimensional.

Lemma 3.2. *Let (V, ω) be a symplectic vector space and let $U \leq V$ be a coisotropic subspace. Then there is a unique symplectic bilinear form ω_0 on U^{ω}/U such that*

$$i^*\omega = q^*\omega_0$$

Proof. First of all, $U \leq U^{\omega}$ since U is coisotropic. Let $v, v \in U^{\omega}$, $u, u' \in U^{\omega}$, and observe that

$$\omega(u + v, u + v') = 0$$

by the bilinearity of ω. Thus ω descends to a unique form ω_0 on U^ω/U. Now suppose that $v \in U^\omega$ and $\omega(v,u) = 0$ for all $u \in U$. It follows that $v \in (U^\omega)^\omega = U$, and hence ω_0 is nondegenerate on U^ω/U. $\square$

3.3 Symplectic Manifolds

Let M be a smooth manifold.

Definition 3.4. A *symplectic structure* on M is a closed 2-form $\omega \in \Omega^2(M)$ which is nondegenerate in the sense that $\iota_X\omega \neq 0$ for all $X \in \mathfrak{X}(M)$. The pair (M,ω) is called a *symplectic manifold*.

In Parts II and III, we will sometimes refer to this construction as a *classical symplectic structure* so as to differentiate it from its vector-valued extension.

Kirillov [41] identifies three primary sources of symplectic manifolds:

1. algebraic submanifolds of the complex projective space $\mathbb{C}P^N$.

2. the coadjoint orbits of a compact semisimple Lie group,

3. the phase space $(T^*Q, -\mathrm{d}\theta)$ of a smooth manifold Q.

The first of these has no clear extension to our eventual vector-valued formalism, and we will not discuss it further in this chapter. The coadjoint orbits are central to the theory of reduction, though the construction is moderately technical and we defer to [41] for three independent constructions of the natural symplectic structure. Phase spaces arise from the theory of dynamics, and historically formed the first examples of symplectic manifolds. We reserve our treatment of these spaces for Chapter 8, where they will fit into a broader class of more general vector-valued examples.

Example 3.3. Equip $\mathbb{R}^{2n}$ with coordinates $(x^i, y^i)_{i\leq n}$ and let

$$\omega = \sum_{i\leq n} \mathrm{d}x^i \wedge \mathrm{d}y^i$$

Since ω is invariant under translations, it follows that it is parallel with respect to the standard metric on $\mathbb{R}^{2n}$ and hence is closed. In terms of the basis (x^i, y^i), ω has the form of the symplectic structures of Example 3.1 and thus ω is nondegenerate. It follows that ω is a symplectic structure on $\mathbb{R}^{2n}$.

Example 3.4. Consider the unit sphere $S^2 \subseteq \mathbb{R}^3$ and define the 2-form $\omega \in \Omega^2(S^2)$ by

$$\omega_v(X_v, Y_v) = \langle v, X \times Y\rangle, \qquad v, X, Y \in \mathbb{R}^3$$

Put differently, $\omega_v(X_v, Y_v)$ is the signed area of the parallelepiped described by X and Y. Since ω is preserved under the group of isometries $\mathrm{Iso}(S^2, g_{\mathbb{R}^3})$ it follows that ω is parallel and hence closed. To see that ω is nondegenerate, let $v \in S^2$ and let $X_v \in T_vS^2$ be nonzero. It follows that $(v \times X)_v \in T_vS^2$ and

$$\omega(X, v \times X) = \|X\|^2 \neq 0$$

Thus ω is a symplectic form on S^2.

A symplectic manifold possesses a canonical volume form.

Definition 3.5. Let (M^{2n}, ω) be a symplectic manifold. The *symplectic volume* of M is the quanitity

$$\operatorname{vol} M = \frac{1}{n!} \int_M \omega^n$$

The $(2n)$-form $\frac{1}{n!}\omega^n \in \Omega^{2n}(M)$ is called the *symplectic volume form.*

Since the integral only depends on the top-level component of $\frac{1}{n!}\omega^n$, the notation

$$\operatorname{vol} M = \int_M e^\omega$$

is occasionally seen. Here we interpret $e^\omega = \sum_{k \geq 0} \frac{1}{k!}\omega^k$ as a polynomial in ω; the series terminates as $\omega^k = 0$ for $k > 2n$.

The symplectic volume form returns the value of 1 when evaluated on any symplectic basis. That is, $\frac{1}{n!}\omega^n(e_1, \ldots, e_n, f_1, \ldots, f_n) = 1$ where $\{e_i, \ldots, e_n, f_1, \ldots, f_n\}$ is a basis corresponding to the columns of the matrix in Example 3.1 with respect to the symplectic structure given by that matrix. Moreover, the symplectic volume form is equal to the metric volume form on a Kähler manifold (M, g, ω).

Definition 3.6. Suppose (M, ω) is a symplectic manifold. The *Hamiltonian vector field* $X_f \in \mathfrak{X}(M)$ of the function $f \in C^\infty(M)$ is the symplectic dual of the 1-form $\mathrm{d}f$.

That is,

$$\mathrm{d}f(Y) = \omega(Y, X_f)$$

for all vector fields $Y \in \mathfrak{X}(M)$.

Definition 3.7. Let (M, ω) be a symplectic manifold. The *Poisson bracket* $\{\,,\}$ on the algebra of functions $C^\infty(M)$ is given by

$$\{f, f'\} = -\omega(X_f, X_{f'})$$

Note that $\{\,,\}$ is both a Lie bracket on and a bi-derivation, that is, a derivation in both arguments, on $C^\infty(M)$.

Definition 3.8. The vector field $X \in \mathfrak{X}(M)$ is called

1. *symplectic* if the flow of X preserves ω,
2. *Hamiltonian* if X is the Hamiltonian vector field for some function $f \in C^\infty(M)$,
3. *locally Hamiltonian* if every point of M has a neighborhood U on which $X|_U$ is Hamiltonian.

Remark 3.3. Note that the assignment

$$\begin{aligned} s\text{-grad} : C^\infty(M) &\longrightarrow \mathfrak{X}(M) \\ f &\longmapsto \{\,, f\} \end{aligned}$$

which sends a function $f \in C^\infty(M)$ to its Hamiltonian vector field $\{\,,f\}$ called the *symplectic gradient*, is an *anti*-homomorphism of Lie algebras. Also note that, as a derivation on the algebra of germs at each point $x \in M$, the expression $\{\,,f\}$ is indeed a vector field on M.

Proposition 3.1. *The vector field $X \in \mathfrak{X}(M)$ is locally Hamiltonian if and only if X is symplectic.*

Proof. Since ω is closed, an application of Cartan's formula yields

$$\mathrm{d}\iota_X\omega = \mathcal{L}_X\omega$$

Consequently, there is a locally-defined function $f \in C^\infty(U)$ with $\mathrm{d}f = \iota_X\omega$ precisely when $\mathcal{L}_X\omega = 0$. $\square$

3.4 Hamiltonian Actions

Definition 3.9. A *comoment map* for the action of G on M is a function $\mu : \mathfrak{g} \to C^\infty(M)$ which satisfies the following two properties

(i) $\tilde{\mu}$ is a lift of the induced map $Y \mapsto \underline{Y}$, that is, $X_{\tilde{\mu}(Y)} = \underline{Y}$.

(ii) $\tilde{\mu}$ is a morphism of Lie algebras, that is,

$$\tilde{\mu}\big([Y,Z]\big) = \big\{\tilde{\mu}(Y), \tilde{\mu}(Z)\big\}$$

We will call $\tilde{\mu}$ a *weak comoment map* if it satisfies condition (i). The *moment map* associated to $\tilde{\mu}$ is the assignment

$$\mu : M \to \mathfrak{g}^*$$

given by interchanging the arguments of $\tilde{\mu}$,

$$\mu(x)(Y) = \tilde{\mu}(Y)(x), \qquad x \in M,\, Y \in \mathfrak{g}$$

The action of G is called *Hamiltonian* (resp. *weakly Hamiltonian*) if it admits a moment map (resp. weak comoment map). In this case, the quadruple (M, ω, G, μ) is referred to as a *Hamiltonian system* (resp. *weakly Hamiltonian system*).

Remark 3.4. Strictly speaking, our Hamiltonian system (M, ω, G, μ) should instead be a quadruple $(M, \omega, \lambda, \mu)$, where λ is an action a Lie group G on M. We will follow the usual conventions and replace λ by the group G. As we will only consider effective actions, the reader who objects to our admittedly informal shorthand is welcome to identify G with its image in Diff M.

Observe that action of G on (M, ω) is weakly Hamiltonian exactly when the induced vector fields $\underline{X} \in \mathfrak{X}(M)$ are Hamiltonian. The additional criterion that $\tilde{\mu}$ be a morphism of Lie algebras is essential to the proof of the reduction theorems to follow.

When we wish to establish that a particular function $\mu : M \to \mathfrak{g}$ is a moment map for a given action of G on (M, ω), we frequently invoke the following equivalent condition.

Proposition 3.2. *Let (M, ω) be a symplectic manifold equipped with a symplectic action of the connected Lie group G. Then the function $\mu : M \to \mathfrak{g}^*$ is a moment map if and only if*

$$\langle \mu_* X, Y \rangle = \omega_x(X, \underline{Y}_x)$$

for all $x \in M$, $X \in T_x M$, and $Y \in \mathfrak{g}$.

Proof. Define the map $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$ by

$$\tilde{\mu}(Y)(f) = \mu(f)(Y)$$

for $Y \in \mathfrak{g}$ and $f \in C^\infty(M)$. Given $X \in T_x M$ and $Y \in \mathfrak{g}$, we have

$$\langle \mu_* X, Y \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\, \mu(x_t) Y \,\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\, \tilde{\mu}(Y)(x_t) \,\Big|_{t=0} = \mathrm{d}\big[\tilde{\mu}(Y)\big](X)$$

where $x_t : I \to M$ is any path tangent to X at 0. It follows that μ is weakly Hamiltonian precisely when

$$\langle \mu_* X, Y \rangle = \mathrm{d}\big[\tilde{\mu}(Y)\big](X) = \omega_x(X, \underline{Y}_x)$$

for all $X \in T_xM$ and $Y \in \mathfrak{g}$.

$\square$

It is readily shown that if in addition $\mu : M \to \mathfrak{g}^*$ is equivariant with respect to the actions of G, then μ is a moment map.

Example 3.5. Equip the unit sphere $S^2 \subseteq \mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ with the symplectic structure of Example 3.4. Let $U(1)$ act on the unit sphere S^2 by rotations about the z-axis, that is,

$$e^{2\pi \mathrm{i} t} \cdot (z, x) = (e^{2\pi \mathrm{i} t} z, x)$$

and define the *height map*,

$$\begin{aligned} \mu : S^2 &\longrightarrow \mathbb{R} \cong \mathfrak{u}(1) \\ (z, x) &\longmapsto x \end{aligned}$$

Then μ is a moment map for the action of $U(1)$ on S^2, with image $[-1, 1] \subseteq \mathbb{R} \cong \mathfrak{u}(1)$.

Example 3.6. Consider the torus $T = \mathbb{R}^2/\mathbb{Z}^2$ with the symplectic structure $\omega \in \Omega^2(T)$ descending from $\mathrm{d}x \wedge \mathrm{d}y$ on $\mathbb{R}^2$. Then the action of $U(1)$ on T given by

$$e^{2\pi \mathrm{i} t} \cdot [x, y] = [t + x, y], \qquad x, y \in \mathbb{R}^2\, g \in$$

it symplectic but not Hamiltonian. In fact, this action does not admit a weak moment map. To see this, observe that any Hamiltonian function $f \in C^\infty(M)$ for the induced vector field $\frac{\partial}{\partial x} \in \mathfrak{X}(T)$ of $1 \in \mathrm{u}(1) \cong \mathbb{R}$ must satisfy the equation $\mathrm{d}f = \mathrm{d}y$, but $\mathrm{d}y \in \Omega^1(T)$ is not exact and so there can be no such function $f \in C^\infty(T)$.

Example 3.7. The *coadjoint action* $\mathrm{Ad}_g^* : G \curvearrowright \mathfrak{g}^*$ of a Lie group G on the dual $\mathfrak{g}^*$ Lie algebra $\mathfrak{g}$ is defined by

$$(g \cdot \alpha)(Y) = \alpha(\mathrm{Ad}_g^{-1} Y)$$

for $g \in G$, $\alpha \in \mathfrak{g}^*$, and $Y \in \mathfrak{g}$. The orbits $\mathcal{O} \subseteq \mathfrak{g}^*$ of this action possess a canonical symplectic structure. Though we decline to outline the construction of this symplectic structure, referring instead to [41], we do note that the inclusion $i : \mathcal{O} \hookrightarrow \mathfrak{g}^*$ is a moment map for the restricted action $\mathrm{Ad}^* : G \curvearrowright \mathfrak{g}^*$.

We record the following proposition which ensures that Hamiltonian actions descend to subgroups in a predictable way.

Proposition 3.3. *Let (M, ω, G, μ) be a Hamiltonian system, let $H \leq G$ be a subgroup of G, and let $i : \mathfrak{h} \to \mathfrak{g}$ be the inclusion of Lie algebras. Then the restriction $\mu|_{\mathfrak{h}} : M \to \mathfrak{h}^*$ where, specifically*

$$\mu|_{\mathfrak{h}}(x) = \mu(x)|_{\mathfrak{h}} \in \mathfrak{h}^*$$

is a moment map for the action of H on M.

Proof. Let $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$ be the comoment map associated to μ. Since $\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$ is a Lie group morphism which lifts the induced vector fields of G in $\mathfrak{X}(M)$ to

$C^\infty(M)$ by the symplectic gradient, it follows that its restriction to any Lie subalgebra $\mathfrak{h} \leq \mathfrak{g}$ is as well. Thus $\tilde{\mu}|_{\mathfrak{h}}$ is a comoment map for the action of H, and

$$\mu|_{\mathfrak{h}}(x)(Y) = \mu(x)(Y) = \tilde{\mu}|_{\mathfrak{h}}(Y)(x), \qquad x \in M,\, Y \in \mathfrak{h}$$

implies that $\mu|_{\mathfrak{h}}$ is the corresponding moment map.

$\square$

We might alternatively denote the map $\mu|_{\mathfrak{h}} : M \to \mathfrak{h}$ by $i^*\mu$, where $i : \mathfrak{h} \to \mathfrak{g}$ is the inclusion. However, whereas the symbol $\mu|_{\mathfrak{h}}$ is clear, the notation $i^*\mu$ is ambiguous as it falsely appears to be a pullback of μ.

Not only is it true that moment maps need not exist, it is also the case that when they do exist they are not always unique. Our next result addresses the structure of the space of moment maps for a Hamiltonian action.

Proposition 3.4. *The set of moment maps (resp. weak moment maps) for a Hamiltonian action $\lambda : G \curvearrowright M$ is an affine space modeled on the annihilator $[\mathfrak{g}, \mathfrak{g}]^0 \leq \mathfrak{g}^*$ (resp. the dual $\mathfrak{g}*$).*

Proof. Let $\mathcal{M}'$ and $\tilde{\mathcal{M}}'$ denote the sets of weak moment and comoment maps, respectively, for the action λ. For $\mu, \nu \in \tilde{\mathcal{M}}'$ and $Y \in \mathfrak{X}(M)$, we have

$$X_{\tilde{\mu}(Y)} = \underline{Y} = X_{\tilde{\nu}(Y)}$$

from which

$$\mathrm{d}\tilde{\mu}(Y) = \iota_{\underline{Y}}\omega = \mathrm{d}\tilde{\nu}(Y)$$

and thus $\tilde{\mu}(Y) = \tilde{\nu}(Y) + C(Y)$ for some constant function $C(Y) \in \mathbb{R}$. As $\tilde{\mu}, \tilde{\nu} : \mathfrak{g}^* \to C^\infty(M)$ are linear, it follows that $C : \mathfrak{g} \to \mathbb{R}$ is linear as well, that is, $C \in \mathfrak{g}^*$. Conversely, for any $\alpha \in \mathfrak{g}^*$,

$$X_{\tilde{\mu}(Y)+\alpha(Y)} = X_{\tilde{\mu}(Y)} = \underline{Y}$$

and thus $\tilde{\mu} + \alpha$ is a weak comoment map for λ. We deduce that $\mathcal{M}' \cong \tilde{\mathcal{M}}'$ is a $\mathfrak{g}^*$-affine space.

Now let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ denote the sets of moment and comoment maps, respectively, for λ and let $\alpha \in \mathfrak{g}^*$. We have established that $\tilde{\mu} + \alpha$ is a weak comoment map. It remains to determine the conditions under which $\tilde{\mu} + \alpha$ preserves the Lie algebra structures of $\mathfrak{g}$ and $C^\infty(M)$. Observe that $\tilde{\mu} + \alpha$ is a comoment map precisely when

$$\begin{aligned}
\tilde{\mu}\big([Y,Z]\big) + \alpha\big([Y,Z]\big) &= \big\{\tilde{\mu}(Y) + \alpha(Y), \tilde{\mu}(Z) + \alpha(Z)\big\} \\
&= \big\{\tilde{\mu}(Y), \tilde{\mu}(Z)\big\} \\
&= \tilde{\mu}\big([Y,Z]\big)
\end{aligned}$$

for all $Y, Z \in \mathfrak{g}$, where the second equality follows as $\alpha(Y)$ and $\alpha(Z)$ are constant functions. We conclude that $\tilde{\mu} + \alpha$ is a comoment map precisely when α vanishes on

$[\mathfrak{g}, \mathfrak{g}] \leq \mathfrak{g}$. □

Corollary 3.1. *If (M, ω, G, μ) is a Hamiltonian system and G is semisimple, then μ is the unique moment map for the action of G on (M, ω).*

We will require the following result for the proof of the symplectic reduction theorem. In particular, we need to ensure that the action of G on M preserves the 0-level set of the moment map $\mu : M \to \mathfrak{g}^*$.

Lemma 3.3. *If G is connected then the moment map μ is G-equivariant.*

Proof. For $Y, Z \in \mathfrak{g}$, we have

$$\begin{aligned}
\tilde{\mu}(\mathrm{ad}_Y Z) &= \big\{\tilde{\mu}(Y), \tilde{\mu}(Z)\big\} \\
&= X_{\tilde{\mu}(Y)}\tilde{\mu}(Z) \\
&= \underline{Y}\ \tilde{\mu}(Z)
\end{aligned}$$

Since G is connected, it follows that $\tilde{\mu}(\mathrm{Ad}_g Z) = g \cdot \tilde{\mu}(Z)$. Now fix $x \in M$ and observe that

$$\begin{aligned}
\mu(g \cdot x)(Y) &= \tilde{\mu}(Y)(g \cdot x) \\
&= \big[g^{-1} \cdot \tilde{\mu}(Y)\big](x) \\
&= \tilde{\mu}(\mathrm{Ad}_g^{-1} Y)(x) \\
&= \big[\mathrm{Ad}_g^* \, \mu(x)\big](Y)
\end{aligned}$$

Thus μ is equivariant. $\square$

If G has multiple components, then we must include the additional condition that the moment map $\mu : M \to \mathfrak{g}^*$ is equivariant. In this dissertation, we will always assume that this is the case.

3.5 Symplectic Reduction

We now present the fundamental theorem of Hamiltonian systems.

Theorem 3.1 (Marsden-Weinstein, Meyers). *Let (M, ω, G, μ) be a Hamiltonian system. If $M_0 = \mu^{-1}(0)/G$ is a manifold, then there is a unique 2-form $\omega_0 \in \Omega^2(M)$ on M_0 such that*

$$\pi^*\omega_0 = i^*\omega$$

where $\pi : \mu^{-1}(0) \to M_0$ is the quotient map, and $i : \mu^{-1}(0) \hookrightarrow M$ is the inclusion. Moreover, the 2-form ω_0 is symplectic.

For clarity, we illustrate below the relevant spaces and forms, as well as the relations between them.

$$\begin{tikzcd}
 & i^*\omega & \omega \\
q^*\omega_0 & \mu^{-1}(0) \arrow[r, hook, "i"] \arrow[d, "q"'] & M \\
\omega_0 & \mu^{-1}(0)/G &
\end{tikzcd}$$

Proof. Suppose M_0 is a manifold. As $\mu : M \to \mathfrak{g}^*$ is equivariant, G preserves the level set $\mu^{-1}(0)$ and thus $\underline{\mathfrak{g}} \leq \ker \mu_*$ on $\mu^{-1}(0)$. Since

$$\begin{aligned} X \in \ker \mu_* &\iff \forall Y \in \mathfrak{g} \, : \, \langle \mu_* X, Y \rangle \\ &\iff \forall Y \in \mathfrak{g} \, : \, \omega(X, \underline{Y}) \\ &\iff X \in \underline{\mathfrak{g}}^\omega \end{aligned}$$

we deduce from Lemma 3.1 that

$$\underline{\mathfrak{g}} = \underline{\mathfrak{g}}^{\omega\omega} = (\ker \mu_*)^\omega = \left[T\mu^{-1}(0)\right]^\omega$$

and we conclude from Lemma 3.2 that $\omega \in \Omega^2\big(\mu^{-1}(0)\big)$ descends to a unique nondegenerate form on $M_0 = \mu^{-1}(0)/G$. ☐

Remark 3.5. It should be emphasized that the reduced space (M_0, ω_0) of a Hamiltonian system (M, ω, G, μ) depends nontrivially on the choice of moment map μ. That is, distinct moment maps can yield inequivalent reduced spaces. It is also possible for the preimage $\mu^{-1}(0)$, and hence the reduction $\mu^{-1}(0)/G$, to be empty.

We note that the designation *reduction* may both to the process by which a Hamiltonian system is rendered a symplectic manifold, as well as to the reduced symplectic manifold itself.

We will at times write $(M/\!/G, \omega_{/\!/G})$ for the reduction of (M, ω, G, μ). This notation is useful when multiple groups are involved; see, for example, Theorem 3.3 below.

The symplectic reduction theorem is occasionally stated in greater generality, as follows.

Theorem 3.2 (Stabilizer Reduction)**.** *Let (M, ω, G, μ) be a Hamiltonian system and fix $\alpha \in \mathfrak{g}^*$. If $M_\alpha = \mu^{-1}(\alpha)/G_\alpha$ is a manifold, where G_α is the* Ad^**-stabilizer of α, then there is a unique 2-form $\omega_\alpha \in \Omega^2(M)$ on M_0 such that $\pi^*\omega_\alpha = i^*\omega$. Moreover, the 2-form ω_α is symplectic.*

For a proof of this result, see, for example, [4, 65] or, less directly, Proposition 3.5 below. With the aim of generalizing to the vector-valued theory of Part II, we present a similar result as a corollary of Theorem 3.1.

Corollary 3.2 (Stabilizer Reduction, II)**.** *Suppose that (M, ω, G, μ) is a Hamiltonian system, fix $\alpha \in \mathfrak{g}^*$, let $G_\alpha \leq G$ be the stabilizer of α under the coadjoint action, and let $\mathfrak{g}_\alpha \leq \mathfrak{g}$ be the Lie algebra of G_α. Put $\mu_\alpha = \mu|_{\mathfrak{g}_\alpha} : M \to \mathfrak{g}_\alpha^*$. When $M_\alpha = \mu_\alpha^{-1}(\alpha)/G_\alpha$ is a manifold, there is a unique 2-form $\omega_\alpha \in \Omega^2(M_\alpha)$ which satisfies $\pi^*\omega_\alpha = i^*\omega$. Moreover, ω_α is symplectic.*

Proof. Proposition 3.3 implies that $\mu|_{\mathfrak{g}_\alpha}$ is a moment map for the action of G_α on M. Since

$$\alpha\,[Y, Z] = (\mathrm{ad}^*_Z \alpha)(Y) = 0, \qquad Y, Z \in \mathfrak{g}_\alpha$$

we have $\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha]^0 \leq \mathfrak{g}^*$. As the inclusion $\mathfrak{g}_\alpha \hookrightarrow \mathfrak{g}$ is a morphism of Lie algebras, it follows that $\alpha|_{\mathfrak{g}_\alpha} \in [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha]^0 \leq \mathfrak{g}_\alpha^*$ and Proposition 3.4 implies that $(\mu - \alpha)|_{\mathfrak{g}_\alpha}$ is a moment map for G_α. The result follows by an application of Theorem 3.1.

□

It follows from the proofs of Theorem 3.2 and 3.2 that

$$\ker \mu_* = \underline{\mathfrak{g}}^\omega = \ker(\mu_\alpha)_*$$

from which we deduce that $\mu^{-1}(0)/G_\alpha = \mu_\alpha^{-1}(0)/G_\alpha$. This equality, which obtains in the classical case, justifies our use of the notation $(M_\alpha, \omega_\alpha)$ to refer to either space.

Since μ is G-equivariant, and since G acts by symplectomorphims, it follows that

$$\mu^{-1}(g \cdot \alpha) = g\mu^{-1}(\alpha) \cong \mu^{-1}(\alpha)$$

as symplectic manifolds when they are smooth. Since $G_{g\alpha} = (G_\alpha)^g$, there is a canonical symplectic identification of reduced spaces $M_\alpha \cong M_{g\alpha}$ given by

$$\begin{aligned} \mu^{-1}(\alpha)/G_\alpha &\xrightarrow{\sim} \mu^{-1}(g \cdot \alpha)/G_{g\alpha} \\ G_\alpha \cdot x &\longmapsto g \cdot (G_\alpha \cdot x) = G_{g\alpha} \cdot gx \end{aligned}$$

In this way, the reduction M_α depends only on the coadjoint orbit $\mathcal{O}_\alpha$. We now present a method for reducing the entire preimage $\mu^{-1}(\mathcal{O}_\alpha)$ by G. We will later show that this reduction is canonically isomorphic to the stabilizer reduction $\mu^{-1}(\alpha)/G_\alpha$

for any $\alpha \in \mathcal{O}_\alpha$.

Corollary 3.3 (Orbit Reduction)**.** *Let (M, ω, G, μ) be a Hamiltonian system and let $\mathcal{O}_\alpha \subseteq \mathfrak{g}^*$ denote a coadjoint orbit. If $M_{\mathcal{O}} = \mu^{-1}(\mathcal{O}_\alpha)/G$ is smooth, then there is a unique 2-form $\omega_{\mathcal{O}} \in \Omega^2(M_{\mathcal{O}})$ such that $\pi^*\omega_{\mathcal{O}} = \mu^*\eta + i^*\omega$ where η is the standard symplectic form on $\mathcal{O}$. Moreover, $\omega_{\mathcal{O}}$ is symplectic.*

Proof. First extend the action of G on M to the natural action of G on $M' = (\mathcal{O}^- \times M, \omega')$, where $\mathcal{O}^-$ denotes the orbit $\mathcal{O}$ equipped with the symplectic structure $-\eta$. Note that the moment map for the action of G on M' is $-\mathrm{id} + \mu$. Since the diffeomorphism

$$\begin{aligned} \phi : \mu^{-1}(\mathcal{O}) &\longrightarrow (-\mathrm{id} + \mu)^{-1}(0) \\ x &\longmapsto \big(\mu(x), x\big) \end{aligned}$$

is equivariant, we deduce that

$$\mu^{-1}(\mathcal{O})/G \cong (\mathcal{O}^- \times M)//G$$

Let $\omega_0 = \phi^*_{/G}\,\omega'_0$ and observe that the equivariance of ϕ yields

$$\begin{aligned}
\pi^*\omega_0 &= \phi^*(\pi'^*\omega'_0) \\
&= \phi^*(i'^*\omega') \\
&= \phi^*\omega' \\
&= \mu^*\eta + \omega
\end{aligned}$$

Since ϕ maps into $(-\mathrm{id} + \mu)^{-1}(0)$, it follows that $i' \circ \phi = \phi$, and the result follows by the equality

$$\pi^*\omega_0 = \phi^*\omega'$$

□

Proposition 3.5. *Let (M, ω, G, μ) be a Hamiltonian system and let $\alpha \in \mathfrak{g}^*$. Then there is a canonical symplectic isomorphism*

$$\mu^{-1}(\alpha)/G_\alpha \cong \mu^{-1}(\mathcal{O}_\alpha)/G$$

Proof. Consider the maps

$$\begin{aligned}
i &: \mu^{-1}(\alpha) \longrightarrow M \\
\pi &: \mu^{-1}(\alpha) \longrightarrow \mu^{-1}(\alpha)/G_\alpha
\end{aligned}$$

on $\mu^{-1}(\alpha)$ and

$$i' : \mu^{-1}(\mathcal{O}_\alpha) \longrightarrow M$$

$$\pi' : \mu^{-1}(O_\alpha) \longrightarrow \mu^{-1}(\alpha)/G$$

on $\mu^{-1}(\mathcal{O}_\alpha)$, and denote the inclusion by

$$j : \mu^{-1}(\alpha) \longrightarrow \mu^{-1}(\mathcal{O}_\alpha)$$

Finally, write $\omega_{0,\alpha}$ and $\omega_{0,\mathcal{O}_\alpha}$ for the reduced forms on $\mu^{-1}(\alpha)$ and $\mu^{-1}(\mathcal{O}_\alpha)$, respectively.

By the equivariance of μ, every G-orbit in $\mu^{-1}(\mathcal{O}_\alpha)$ has a representative in $\mu^{-1}(\alpha)$. We thus obtain a morphism between G_α-principal bundles π and $\pi' \circ j$, given by

$$\begin{tikzcd}
\mu^{-1}(\alpha) \arrow[r, "\mathrm{id}"] \arrow[d, "\pi"'] & \mu^{-1}(\alpha) \arrow[d, "\pi' \circ j"] \\
\mu^{-1}(\alpha)/G_\alpha \arrow[r, "\sim"] & \mu^{-1}(\mathcal{O}_\alpha)/G \\
G_\alpha \cdot x \arrow[r, mapsto] & G \cdot x
\end{tikzcd}$$

Applying the action of G_α to the equality

$$i^*\omega = j^*(\mu^*\eta + \omega) = (\pi' \circ j)^*\omega_{0,\mathcal{O}_\alpha}$$

we conclude that $\omega_{0,\alpha} = \omega_{0,\mathcal{O}_\alpha}$. $\square$

Theorem 3.3 (Reduction by Stages). *Let $\phi : K \to \mathrm{Aut}H$ be a Lie group morphism and suppose that the semidirect product $G = H \rtimes_\phi K$ acts on M in a Hamiltonian fashion. Then K acts on $M//H$ and*

$$M//(H \rtimes_\phi K) = (M//H)//K$$

Proof. As manifolds, $G = H \times K$ and thus any corresponding moment map is of the form

$$\mu + \nu : M \to \mathfrak{h}^* + \mathfrak{k}^*$$

Consider the maps

$$\begin{aligned} \tilde{\mu} : \mathfrak{h} &\longrightarrow C^\infty(M) \\ X &\longmapsto \mu(\,\cdot\,)(X) \end{aligned}$$

and

$$\begin{aligned} \tilde{\nu} : \mathfrak{k} &\longrightarrow C^\infty(M) \\ X &\longmapsto \nu(\,\cdot\,)(X) \end{aligned}$$

and observe that the comoment map associated to $\mu + \nu$ is given by

$$\begin{aligned}\widetilde{\mu+\nu}\,(Y,Z)(x) &= (\mu+\nu)(x)(Y,Z) \\ &= \mu(x)(Y) + \nu(x)(Z) \\ &= \tilde{\mu}(Y)(x) + \tilde{\nu}(Z)(x) \\ &= (\tilde{\mu}+\tilde{\nu})(Y,Z)(x)\end{aligned}$$

Since $\tilde{\mu}+\tilde{\nu} : \mathfrak{h}+\mathfrak{k} \to C^\infty(M)$ is a comoment map, it follows that the restrictions $\tilde{\mu} : \mathfrak{h} \to C^\infty(M)$ and $\tilde{\mu} : \mathfrak{k} \to C^\infty(M)$ are comoment maps as well. We conclude that μ and ν are moment maps for the actions of H and K, respectively.

Since $\mu+\nu$ is G-equivariant, it follows that the subgroup K preserves the set $(\mu+\nu)^{-1}\mathfrak{k} = \mu^{-1}(0)$. Moreover, as

$$k \cdot (H \cdot x) = \phi_k(H) \cdot (k \cdot x) = H \cdot (k \cdot x)$$

we deduce that the action of K descends to $\mu^{-1}(0)/H = M//H$.

It remains to show that $(M//H)//K = M//(H \rtimes K)$. Since $\mu+\nu$ is G-equivariant, and since K fixes the subspace $\mathfrak{h}^* \leq \mathfrak{h}^* + \mathfrak{k}^*$, it follows that ν descends to a map $\nu_{/H} : M/H \to \mathfrak{k}^*$. Restriction to $\mu^{-1}(0)/H$ yields a map $\nu_{//H} : M//H \to \mathfrak{k}$. Since $\nu_{//H}$ and $\omega_{//H}$ are defined in terms of their action on representatives, it follows that

$$\big\langle (\nu_{//H})_* X, Y \big\rangle = \omega_{//H}(X, \underline{Y})$$

for all $X \in T(M//H)$ and $Y \in \mathfrak{k}$. Thus $\nu_{//H}$ is a moment map for the action of K on $M//H$. Using the fact that K acts on M/H, we obtain

$$\begin{aligned}(\mu+\nu)^{-1}(0)/(H \rtimes K) &= \left[\mu^{-1}(0)/H \,\cap\, \nu^{-1}(0)/H\right]/K \\ &= \left[M//H \,\cap\, \nu_{/H}^{-1}(0)\right]/K \\ &= \nu_{//H}^{-1}(0)/K\end{aligned}$$

and thus

$$M//(H \rtimes K) = (M//H)//K$$

as required. □

3.6 The Topology of the Reduced Space

We turn very briefly to the case in which $M_0 = \mu^{-1}(0)/G$ is not assumed to be smooth. The investigation of singular reductions has been an active area of research and there is much to be said, though this is not the primary line of development that we take in this work. Our approach will be to present an interesting and well-known result in the singular case, and then mention the more general theory.

First we establish a technical lemma.

Lemma 3.4. *Let (M, ω, G, μ) be a Hamiltonian system. The following are equivalent,*

(i) $0 \in \mathfrak{g}^*$ *is a regular value of* $\mu : M \to \mathfrak{g}^*$,

(ii) The stabilizer $G_x \leq G$ is discrete for every $x \in \mu^{-1}(0)$.

If G is compact, then we may replace discrete *with* finite *in Condition (ii).*

Proof. First suppose that $x \in \mu^{-1}(0)$ is a singular point of μ, so that $\mu_* : T_xM \to \mathfrak{g}^*$ is not surjective. Thus the annihilator $(\mu_* T_xM)^0 \leq \mathfrak{g}$ is nontrivial and there is a $Y \in \mathfrak{g}$ with

$$\omega_x(T_xM, \underline{Y}_x) = \langle \mu_* T_xM, Y \rangle = 0$$

As ω_x is nondegenerate, we infer that $\underline{Y}_x$ is zero and, consequently, that Y is tangent to G_x, which is therefore continuous.

We trace back our steps to obtain the reverse implication. The assertion regarding compact G is clear. □

We immediately obtain the following theorem.

Theorem 3.4. *Let (M, ω, G, μ) be a Hamiltonian system with compact Lie group G, and suppose that $0 \in \mathfrak{g}^*$ is a regular value of the moment map $\mu : M \to \mathfrak{g}^*$. Then M_0 has at most orbifold singularities.*

Proof. By Lemma 3.4, the group G acts on $\mu^{-1}(0)$ with finite stabilizers, and thus the quotient $\mu^{-1}(0)/G$ is an orbifold. □

In the general case, in which we suppose only that G is compact, the reduction of (M, ω, G, μ) is shown to have at most conic singularities [65]. In [80] it is shown that M_0 is a *stratified symplectic space*, that is, M_0 admits a structured decomposition into symplectic manifolds which refines the orbit-type decomposition of M_0.

Chapter 4

The Space of Connections on a Surface

This short chapter, which forms a synthesis of the previous two, is independent from those to follow. We include it here both for its intrinsic interest and as an exposition of the classical material which is to form the catalyst for the later development of this work.

Let Σ denote a closed compact oriented surface, let G be a compact Lie group, and let E be the trivial vector bundle on Σ with fiber V. Combining the theories of curvature and symplectic geometry, we will show that the space of connections $\mathcal{A}(E)$ on E inherits the structure of an infinite-dimensional symplectic manifold with symplectic form $\omega \in \Omega^2\big(\mathcal{A}(P)\big)$, given by

$$\omega_A(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta \tag{$*$}$$

for $A \in \mathcal{A}(P)$ and $\alpha, \beta \in \Omega^1(\Sigma, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$. Moreover, we will show that the action of the gauge group $\mathcal{G}$ on $\mathcal{A}(E)$ is Hamiltonian, with moment map induced by the curvature

$$R : \mathcal{A}(E) \to \Omega^2(\Sigma, \mathrm{ad}E)$$

We refer to Part 3 for the treatment of an arbitrary principal bundle P on Σ, which will arise as a special case of more general constructions.

4.1 Cohomology as Symplectic Reduction

We will take as our manifold the space of 1-forms $\Omega^1(\Sigma)$. Note that there is a canonical isomorphism

$$T_A\Omega^1(\Sigma) \cong \Omega^1(\Sigma), \qquad A \in \Omega^1(\Sigma)$$

and that

$$\omega_A(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta$$

defines a symplectic structure on $\Omega^1(\Sigma)$. Consider the group $\big(C^\infty(\Sigma), +\big)$ of smooth functions on Σ under the operation of addition. Let $f \in C^\infty(\Sigma)$ act on $A \in \Omega^1(\Sigma)$ by

$$f \cdot A = \mathrm{d}f + A$$

and observe that induced vector fields of this action are given by

$$\frac{\mathrm{d}}{\mathrm{d}t} tf \cdot A\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t} t\mathrm{d}f + A\Big|_{t=0} = \mathrm{d}f \in T_A\Omega(\Sigma)$$

for $f \in C^\infty(\Sigma)$ and $A \in \Omega^1(\Sigma)$.

Further observe that we may identify the smooth dual $\mathscr{g}^*$ of the Lie algebra $\mathscr{g} = C^\infty(M) = \Omega^0(M)$ with the space $\Omega^2(\Sigma)$ of 2-forms by assigning to each $\eta \in \Omega^2(\Sigma)$ the covector given by

$$f \mapsto \int_\Sigma f\eta, \qquad f \in C^\infty(\Sigma)$$

An application of Stokes' theorem yields

$$\int_\Sigma \mathrm{d}\alpha \wedge f = \int_\Sigma \alpha \wedge \mathrm{d}f$$

which, under the identification $\mathscr{g}^* = \Omega^2(M)$, implies that

$$\langle \mathrm{d}_*\alpha, f\rangle = \langle \mathrm{d}\alpha, f\rangle = \omega_A(\alpha, \underline{f}_A)$$

for $A \in \Omega^1(\Sigma)$, $\alpha \in \Omega^1(\Sigma) \cong T_A\Omega^1(\Sigma)$, and $f \in C^\infty(\Sigma)$, where the first equality follows as the exterior derivative is a linear map. We conclude that the exterior derivative $\mathrm{d} : \Omega^1(M) \to \Omega^2(M)$ is a moment map for the action of $\mathcal{G} = C^\infty(M)$ on $\Omega^1(M)$. Since $\mathcal{G}$ acts via translations of $\Omega^1(\Sigma)$ by coboundaries $\mathrm{d}f \in B^1(\Sigma)$, the reduced space is

$$\Omega^1(\Sigma)_0 = \mu^{-1}(0)/G = Z^1(\Sigma)/B^1(\Sigma) = H^1(\Sigma)$$

where $Z^1(\Sigma)$ is the space of cocycles on $\Omega^1(\Sigma)$.

The reduced symplectic structure ω_0 on $H^1(\Sigma)$ can be expressed in terms of the $H^1(\Sigma)$-module structure of $H_1(\Sigma)$ by

$$\omega_{0,[A]}\big([\alpha],[\beta]\big) = [\alpha] \cap [\beta] \cap [\Sigma] \in H_0(\Sigma) \cong \mathbb{R}$$

for $[A] \in H^1(\Sigma)$ and $[\alpha],[\beta] \in T_AH^1(\Sigma)$, and where $\Sigma \in H_2(\Sigma,\mathbb{Z}) \hookrightarrow H_2(\Sigma)$ is the fundamental class and $\cap$ denotes the cap product. See [29] or [85] for a comprehensive overview of these topics.

4.2 The Reduction of the Space of Connections on a Surface

Let $E = \Sigma \times V$ be the trivial vector bundle over Σ modeled on V, let $G \leq \mathrm{Aut}(V)$, and let $\mathcal{A}_G(E)$ denote the space of G-connections on E. Fix an Ad-invariant metric $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$.

In light of the canonical identification $T_\nabla \mathcal{A}_G(E) \cong C^\infty(\Sigma, \mathfrak{g})$ of Proposition 2.2, there is a natural symplectic structure ω on $\mathcal{A}_G(E)$, given by

$$\omega_\nabla(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta$$

for $\nabla \in \mathcal{A}_G(E)$, $\alpha, \beta \in \Omega^1(\Sigma, \mathfrak{g}) \cong T_\nabla \mathcal{A}_G(E)$. Note that we define

$$\wedge : \Omega^*(\Sigma, \mathfrak{g}) \otimes \Omega^*(\Sigma, \mathfrak{g}) \xrightarrow{\wedge_{\Omega^*(\Sigma)}} \Omega^*(\Sigma, \mathfrak{g} \otimes \mathfrak{g}) \xrightarrow{\langle , \rangle_{\mathfrak{g}}} \Omega^*(\Sigma)$$

Since the exterior derivative is itself a connection on E, Proposition 2.2 also implies that the connections $\nabla \in \mathcal{A}_G(E)$ on E are precisely the operators

$$\nabla = \mathrm{d} + A$$

for $A \in \Omega^1(\Sigma, \mathfrak{g})$. Since E is trivial, the gauge group $\mathcal{G}E$ is equivalent to

$$\mathcal{G} = C^\infty(\Sigma, G)$$

and the Lie algebra and dual Lie algebra of $\mathcal{G}E$ are naturally identified with

$$\mathscr{g} = C^\infty(\Sigma, \mathfrak{g}) = \Omega^0(\Sigma, \mathfrak{g})$$
$$\mathscr{g}^* = \Omega^2(\Sigma, \mathfrak{g})$$

This second identification arises by identifying the form $\eta \in \Omega^2(\Sigma)$ with the assignment $X \mapsto \int_\Sigma \eta \wedge X$. The induced action of $\mathcal{G}$ on $\mathcal{A}_G(E)$ is given by

$$g \cdot \nabla = g \nabla g^{-1}$$

We will show that this action is Hamiltonian.

Lemma 4.1. *If $\nabla = \mathrm{d} + A \in \mathcal{A}_G(E)$, $\alpha \in \Omega^1(\Sigma, \mathfrak{g}) \cong T_\nabla \mathcal{A}_G(E)$, and $X \in C^\infty(\Sigma) \cong \mathscr{g}$, then*

(i)

$$R_* \alpha = \mathrm{d}\alpha + [\alpha, A]$$

(ii)

$$\underline{X}_\nabla = \mathrm{d}X + [X, A]$$

Proof. (i) Recall that

$$R_{\mathrm{d}+A} = \mathrm{d}A + A \wedge A$$

so that

$$R_{\nabla + t\alpha} = \mathrm{d}A + A^2 + t\big[\mathrm{d}\alpha + \alpha \wedge A + A \wedge \alpha\big] + t^2 \alpha \wedge \alpha$$

and thus

$$R_* \alpha = \mathrm{d}\alpha + [\alpha, A]$$

(ii) Let g_t be a path in $\mathcal{G} = C^\infty(\Sigma, G)$ which is tangent to X at $t = 0$. Then

$$\begin{aligned} \underline{X}_{\mathrm{d}+A} &= \frac{\mathrm{d}}{\mathrm{d}t}\, g_t(\mathrm{d} + A) g_t^{-1}\big|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\, g_t \mathrm{d} g_t^{-1} + g_t A g_t^{-1}\big|_{t=0} \end{aligned}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\, g_t A g_t^{-1}\big|_{t=0} = [X, A]$$

and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\, g_t \mathrm{d} g_t^{-1}\big|_{t=0}\, s &= \frac{\mathrm{d}}{\mathrm{d}t}\, g_t \mathrm{d}(g_t^{-1} s)\big|_{t=0} \\ &= X(\mathrm{d}s) - \mathrm{d}(Xs) \\ &= \mathrm{d}(Xs) \end{aligned}$$

we conclude that

$$X_{\mathrm{d}+A} = \mathrm{d}X + [X, A]$$

$\square$

Theorem 4.1. *The curvature map*

$$R : \mathcal{A}_G(E) \longrightarrow \Omega^2(\Sigma, \mathfrak{g}) \cong \mathscr{g}^*$$

$$\mathrm{d} + A \longmapsto \mathrm{d}A + A \wedge A$$

is a moment map for the action of $\mathcal{G}E$ on $\mathcal{A}_G(E)$, and the reduced space is the moduli space $\mathcal{M}_G(E)$ of flat connections on E.

Proof. Fix $\nabla = \mathrm{d} + A$, $\alpha \in \Omega^1(\Sigma, \mathfrak{g}) \cong T_\nabla \mathcal{A}_G(\Sigma)$, and $X \in \Omega^0(\Sigma, \mathfrak{g}) \cong \mathscr{g}$. A direct computation yields

$$\begin{aligned} \int_\Sigma \alpha \wedge \underline{X}_\nabla &= \int_\Sigma \alpha \wedge \big(\mathrm{d}X + [X, A]\big) \\ &= \int_\Sigma \big(\mathrm{d}\alpha + [\alpha, A]\big) \wedge X \\ &= \int_\Sigma R_* \alpha \wedge X \end{aligned}$$

That is,

$$\langle R_* \alpha, X \rangle = \omega_\nabla(\alpha, \underline{X}_\nabla)$$

and we deduce that R is a moment map. It immediately follows that the reduced space $\mathcal{A}_G(M)_0$ is equal to $R^{-1}(0)/\mathcal{G} = \mathcal{M}_G(E)$. $\square$

Chapter 5

Variations on the Classical Formalism

Before proceeding to our exposition of vector-valued symplectic geometry, we first survey a few variations of the classical formalism that have already appeared in the literature.

Among the omissions are *Dirac structures*, which interpolate the role of a Poisson bracket and a symplectic form [13]; *contact* and *cosymplectic manifolds*, which may be interpreted as odd-dimensional analogues of symplectic manifolds, and have applications to physics [9]; *cylinder-valued moment maps*, taking values in certain holonomy-induced lattices of the dual Lie algebra $\mathfrak{g}^*$ and satisfying the property that, unlike the classical moment map, they always exist [12, 72]; and the theory of *multi-moment maps*, which treats compatible group actions preserving closed and nondegenerate 3-forms $c \in \Omega^3(M)$ [55]. Given the duration of the period spanning the advent of symplectic geometry at the start of the eighteenth century to the present day, and in light of the practical and mathematical applications of the material, there has been a substantial amount of work closely related to, but not properly contained in, the classical theory. Indeed, the ostensibly complete reference on moment maps and reduction [72] dedicates a full twenty-three pages to listing those topics that it will *not* discuss.

It is worth noting that all of the theories mentioned above, as well as all that are to follow, possess an appropriately-adapted version of the Hamiltonian reduction theorem. This reinforces the essential status of reduction processes to symplectic geometry.

5.1 Hyperkähler Manifolds

Being equipped with complex-valued symplectic structures, hyperkähler manifolds provides the most proximate extension to our vector-valued theory. Our main reference for this section is [33].

Definition 5.1. A *Quaternionic manifold* (M, I, J, K) is a smooth manifold M equipped with three almost-complex structures $I, J, K \in \mathrm{End}(TM)$ such that $IJK = -1$. If I, J, K are complex structures, then (M, I, J, K) is called a *hypercomplex manifold.*

Quaternionic manifolds are so named as the almost complex structures I, J, K endow the fibers of TM with the structure of a quaternionic vector space. As such, the tangent fibers, and hence the underlying manifold M, are necessarily of dimension divisible by 4. As an admissible pairing of a complex and Riemannian structure yields a Kähler structure, so too does a hypercomplex and Riemannian structure yield a hyperkähler manifold.

Definition 5.2. A *hyperkähler manifold* is a Riemannian manifold $\big(M, \langle\,,\rangle\big)$ equipped with three parallel, metric-compatible complex structures $I, J, K \in \mathrm{Aut}(TM)$ such that $IJK = -1$.

By defining the 2-form $\omega \in \Omega^2(M, \mathbb{C})$ to be

$$\omega(X, Y) = \langle JX, Y\rangle + \mathrm{i}\langle KX, Y\rangle$$

we obtain a complex-valued symplectic form on M which is holomorphic with respect to the complex structure I. Naturally, by a complex valued symplectic structure, we mean a closed 2-form $\omega \in \Omega^2(M, \mathbb{C})$ which is nondegenerate in the sense that $\iota_X\omega \in \Omega^1(M, \mathbb{C})$ is nonzero for all $X \in \mathfrak{X}(M)$. Conversely, it follows from the Calabi-Yau theorem that a compact Kähler manifold $(M, I, \langle\,,\rangle)$ equipped with a holomorphic symplectic structure $\omega \in \Omega^2(M, \mathbb{C})$ induces a unique hyperkähler structure (M, I, J, K) on M [91]. When we wish to emphasize role of ω, we occasionally speak of *holomorphic symplectic manifolds.*

Hitchin [33] identifies two primary sources of hyperkähler manifolds:

1. *twistor theory*, a transference of the hyperkähler formalism to classical holomorphic geometry;

2. *hyperkähler quotients*, an analogue of symplectic reduction to the hyperkähler context.

Hyperkähler manifolds are comparatively sparse. These include coadjoint orbits of complex Lie groups, K3 surfaces of Kähler type [36], various moduli spaces arising from physics [102], as well as cotangent spaces over certain complex manifolds [22]. Hyperkähler structures have also been linked to the theory of supersymmetry from mathematical physics [32].

5.2 Multisymplectic Geometry

Multisymplectic geometry generalizes the theory of symplectic manifolds (M, ω) by relaxing the requirement that the degree of $\omega \in \Omega^*(M)$ be 2.

Definition 5.3. Fix a smooth manifold M and an integer $k \geq 1$. A *k-plectic structure* on M is a closed k-form $\omega \in \Omega^k(M)$ which is nondegenerate in the sense that $\iota_X \omega \neq 0$ for all $X \in \mathfrak{X}(M)$. More generally, the form $\omega \in \Omega^*(M)$ is said to be a *multisymplectic structure* if ω is k-plectic for some $k \geq 1$.

A multisymplectic structure is thus nothing more than a closed and nondegenerate differential form on a smooth manifold M. When M is of dimension $n+1$, the 1-plectic structures $\omega \in \Omega^2(M)$ on M are precisely the symplectic structures on M; the n-plectic structures coincide with the volume forms.

Example 5.1. Let G be a semisimple Lie group with Maurer-Cartan form $\theta \in \Omega^2(G, \mathfrak{g})$ and Killing metric $\langle\,,\rangle$, which may be indefinite. The Cartan 3-form $\chi = \frac{1}{12}\big\langle \theta, [\theta, \theta] \big\rangle \in \Omega^3(G)$ is a 2-plectic form on G.

Example 5.2. Let Q be a smooth manifold of dimension n, fix a positive integer $k \leq n$, and define the *multicotangent bundle* to be $\Lambda^k T^* Q$. The *canonical k-form* $\theta \in \Omega^k(\Lambda^k)$ is given by

$$\theta_\alpha(X_i)_{i \leq k} = \alpha(\pi_* X_i)_i, \qquad X \in T_\alpha \Lambda^k T^* Q$$

The exterior derivative $\omega = -\mathrm{d}\theta \in \Omega^{k+1}(\Lambda^k T^* Q)$ is a k-plectic structure on $\Lambda^k T^* Q$.

Multisymplectic structures abound. In fact, given a smooth manifold M of dimension at least 7, for each integer k between 3 and $n-2$, every cohomology class $H^k(M)$ contains a multisymplectic representative [64].

We note that there is a companion Lagrangian theory as well as a theory of Hamiltonian dynamics. In the multisymplectic realization of Hamiltonian actions, the moment map

$$\mu : M \to \mathrm{Hom}(\mathfrak{g}, \Lambda^{k-1} T^* M)$$

takes values in a bundle on M [71]. Note that $\Lambda^{k-1}T^*M$ is the constant bundle $\mathbb{R}$ when $k = 1$ and M is connected. In this case, $\mathrm{Hom}(\mathfrak{g}, \Lambda^{k-1}T^*M) = \mathfrak{g}^*$ and the construction exhibits the classical theory.

The roots of multisymplectic geometry can be traced back to the late nineteenth-century work of Volterra on extensions of Hamilton's equations to higher-dimensional systems, and first appeared in their modern form in 1973 in two papers, on classical field theory [39] and the calculus of variations in multiple variables [26]. We refer to [31] for further historical remarks. A major application of the multisymplectic formalism is to provide a finite-dimensional Hamiltonian counterpart for certain Lagrangian variational problems which otherwise yield infinite-dimensional Hamiltonian systems under the classical framework [74].

5.3 Lie-Group Valued Moment Maps

The theory of Lie-group valued moment maps begins with the action of a Lie group G on a manifold M and a smooth equivariant map $\mu : M \to G$ and, from this point, imposes the requisite compatibility conditions on a 2-form $\omega \in \Omega^2(M)$ to obtain a quadruple (M, ω, G, μ) which evokes a classical Hamiltonian system.

We note that this section is based on [1], [66], and [72]. The fundamental objects of study in this theory are the quasi-Hamiltonian systems which are defined as follows.

Definition 5.4. A *quasi-Hamiltonian manifold* is a quadruple (M, ω, G, μ) where

G acts on M, the form $\omega \in \Omega^2(M)$ is G-invariant, and $\mu : M \to G$ is an smooth G-equivariant map, such that

(i) $\mathrm{d}\omega = -\mu^*\chi$

(ii) $\iota_{\underline{Y}}\omega = \frac{1}{2}\mu^*\langle \theta + \bar{\theta}, Y\rangle$

(iii) $\ker \omega \cap \ker \mu_* = 0$

Here θ and $\bar{\theta}$ are the left and right Maurer-Cartan forms on G, respectively, we suppose $\langle\, ,\rangle$ is an Ad-invariant metric on the Lie algebra $\mathfrak{g}$, and $\chi = \frac{1}{12}\big\langle \theta, [\theta, \theta]\big\rangle \in \Omega^3(G)$ is corresponding the Cartan 3-form on G.

Perhaps the most natural examples of a quasi-Hamiltonian manifolds are the conjugacy classes of Lie groups.

Example 5.3. Consider a Lie group G, with Ad-invariant metric $\langle\, ,\rangle$, acting on a fixed conjugacy class $\mathcal{C} \subseteq G$ by conjugation. The inclusion $\mu : \mathcal{C} \hookrightarrow G$ provides a G-valued moment map and the associated 2-form $\omega \in \Omega^2(G)$ is given by

$$\omega_g(\underline{Y}, \underline{Z}) = \frac{1}{2}\Big(\langle Z, \mathrm{Ad}_g Y\rangle - \langle Y, \mathrm{Ad}_g Z\rangle\Big)$$

for $Y, Z \in \mathfrak{g}$.

The motivation behind quasi-Hamiltonian manifold has its roots in early attempts to construct a finite-dimensional symplectic G-manifold $\bar{\mathcal{M}}$, containing the space of homomorphism $\mathrm{Hom}(\pi_1 M, G)$ of flat connections on a principal bundle P over a surface Σ, with moment map related to the holonomy map, in such a way that the symplectic reduction of $\bar{\mathcal{M}}$ yields the character variety $\mathrm{Hom}(\pi_1 M, G)/G \cong \bar{\mathcal{M}}$ with the canonical symplectic form ω_0 [35, 37, 66]. Complications relating to the definition of this moment map required that $\bar{\mathcal{M}}$ be noncompact, precluding the application of much of the classical Hamiltonian theory. By taking the moment map to be instead an extension of the holonomy itself, the space $\mathcal{M}$ can be taken to be compact [1].

5.4 Bihamiltonian Systems

A bihamiltonian system encodes a special 2-parameter family of Poisson structures on a smooth manifold M.

Definition 5.5. A *bihamiltonian system* $\big(M, \{\,,\}_0, \{\,,\}_1\big)$ is a manifold M equipped with two Poisson brackets, $\{\,,\}_0$ and $\{\,,\}_1$, which are compatible in the sense that every nonzero linear combination

$$c_0\{\,,\}_0 + c_1\{\,,\}_1, \qquad c_0, c_1 \in \mathbb{R}$$

is a Poisson bracket on $C^\infty(M)$.

The name may be understood from the notion of bihamiltonian manifolds as by a single fixed Poisson bracket $\{\,,\}$ on M and two functions $f_0, f_1 \in C^\infty(M)$ which determine the same Hamiltonian vector field $X \in \mathfrak{X}(M)$, called a *bihamiltonian vector field* [23].

Bihamiltonian systems were introduced, in the language of the "two-fold Hamiltonian equations", in [56] for the purpose of analyzing of the Korteweg-de Vries (KdV) equation, a nonlinear wave equation of general interest. They have since evolved into a general method for approaching integrable Hamiltonian systems [10].

To prefigure material from the following chapters, we remark that a bihamiltonian system $\big(M, \{\,,\}_0, \{\,,\}_1\big)$ may be alternatively considered as an $\mathbb{R}^2$-valued bi-derivation,

$$\begin{aligned} \{\,,\} : C^\infty(M) \times C^\infty(M) &\longrightarrow C^\infty(M, \mathbb{R}^2) \\ \{f, f'\} &\longmapsto \{f, f'\}_0 \oplus \{f, f'\}_1 \end{aligned}$$

such that

$$c \cdot \{\,,\} = c_1\{\,,\}_0 + c_2\{\,,\}_1$$

is Poisson bracket for all nonzero $c = (c_1, c_2) \in \mathbb{R}^2$.

Part II

Vector-Valued Symplectic Geometry

Though our motivation arises from the moduli space of flat connections, we develop the theory of vector-valued symplectic geometry entirely independently. It is not until Part III that we return to apply the vector-valued symplectic formalism to the space of connections.

This part comprises three chapters. Chapter 6 introduces the local theory, that of vector-valued symplectic vector spaces. This chapter is mainly technical and primarily serves to support more interesting results on the global theory to follow. Chapter 7 explores the consequences of extending the space of coefficients of a classical symplectic structure. From the symplectic form, to Hamiltonian vector fields and actions, to the moment map and the reduced space; the parallel theory to Chapter 3 is delineated in Chapter 7. Finally, in Chapter 8 we return to the historical roots of symplectic geometry, and investigate the vector-valued counterpart of classical mechanics. See Section 1.1 for a detailed outline of the main results of this part.

A word on notation. When describing vector-valued symplectic vector spaces, we will denote the base by U and the space of coefficients by V. We will carry this convention over to the global theory and consider V-valued symplectic forms on a manifold M modeled on a vector space U. See Section 2.1 for background on Banach manifolds. To avoid confusion, we will usually denote tangent vectors to M by X, and elements of the Lie algebra $\mathfrak{g}$ by Y and Z.

Chapter 6

Vector-Valued Symplectic Vector Spaces

In this chapter, we introduce the key entity of vector-valued symplectic geometry: the vector-valued symplectic vector space. Our treatment begins with the rudimentary definitions and culminate in a reduction theorem. Thus, this chapter is the vector-valued counterpart of Section 3.2, though we shall see that there is a richer structure in the present context than in the classical case.

Throughout this exposition U and V will denote real vector spaces of differing roles. The space U will represent the underlying space on which a vector-valued form ω is defined, while V represents the space of coefficients. This notation is consistent with that of the following chapters, where we consider manifolds modeled on U and vector-valued forms with coefficients in V.

6.1 Introduction and First Results

The fundamental construction of this chapter is as follows.

Definition 6.1. A *V-valued symplectic vector space* is a pair (U, ω) where U is a vector space and $\omega : \Lambda^2 U \to V$ is an alternating V-valued bilinear form on U which is nondegenerate as a bilinear map, that is, if $u \in U$ with $\iota_u \omega = \omega(u, \cdot) = 0$ then $u = 0$. The map ω is called a *V-valued symplectic form* or *V-valued symplectic structure* on U.

To illustrate the subsequent development, we will consider three recurring examples:

1. An even-dimensional vector space V equipped with the direct sum $\oplus_i \omega_i$ of symplectic structures ω_i,
2. The euclidean space $\mathbb{R}^3$ with the cross product $\times$,
3. A centerless Lie algebra $\mathfrak{g}$ with Lie bracket $[\,,]$.

Example 6.1. Every symplectic vector space in the classical sense is an $\mathbb{R}$-valued symplectic vector space in the vector-valued sense. More generally, let $(U, \omega_i)_{i \leq N}$ be a family of symplectic structures on U and define the $\mathbb{R}^N$-valued map $\oplus_i \omega_i : \Lambda^2 U \to \mathbb{R}^N$ by

$$(\oplus_i \omega_i)(X, Y) = \big(\omega_i(X, Y)\big)_{i \leq N}$$

Then $(U, \oplus_i \omega_i)$ is an $\mathbb{R}^N$-valued symplectic vector space.

The requirement that ω must be alternating and bilinear implies that the dimension of U is at least 2. However, in contrast to the classical case, $\dim U$ is not necessarily even, as the following example shows.

Example 6.2. The cross product, $\times$, is an $\mathbb{R}^3$-valued symplectic structure on $\mathbb{R}^3$. The map $\times$ is an alternating and bilinear. To see that it is nondegenerate, fix an arbitrary $X \in \mathbb{R}^3\backslash\{0\}$ and take $Y \in \mathbb{R}^3\backslash\{0\}$ with $X \perp Y$. It follows that $\|X \times Y\| = \|X\| \cdot \|Y\| > 0$ whence $X \times Y \neq 0$.

Recall that $(\mathbb{R}^3, \times)$ can be identified with the Lie algebra $\mathfrak{so}(3)$. To generalize this example, we recall the following definition.

Definition 6.2. The *center* $\mathfrak{z}$ of a Lie algebra $\big(\mathfrak{g}, [\,,]\big)$ is the ideal

$$\mathfrak{z} = \big\{X \in \mathfrak{g} \,\big|\, [X, \cdot\,] = 0\big\}$$

If $\mathfrak{z} = 0$, then $\mathfrak{g}$ is said to be *centerless.*

We see that $\mathfrak{z}$ is an ideal as it is evidently a linear subspace and, if $X \in \mathfrak{z}$ and $Y \in \mathfrak{g}$, then $[X, Y] = 0 \in \mathfrak{z}$. We thus obtain a quotient Lie algebra $\big(\mathfrak{g}/\mathfrak{z}, [\,,]_{\mathfrak{g}/\mathfrak{z}}\big)$ which, as we will show, provides another class of examples of vector-valued symplectic vector spaces.

Example 6.3. Suppose that $\big(\mathfrak{g}/\mathfrak{z}, [\,,]_{\mathfrak{g}/\mathfrak{z}}\big)$ is the quotient of a Lie algebra $\big(\mathfrak{g}, [\,,]\big)$ by its center $\mathfrak{z}$. The quotient Lie bracket $[\,,]_{\mathfrak{g}/\mathfrak{z}}$ is a $\mathfrak{g}/\mathfrak{z}$-valued symplectic structure on $\mathfrak{g}/\mathfrak{z}$.

To see that $[\,,]_{\mathfrak{g}/\mathfrak{g}}$ is nondegenerate, let $X \in \mathfrak{g}/\mathfrak{z}$ be arbitrary and suppose $[X,\,\cdot]_{\mathfrak{g}/\mathfrak{g}} = 0$ so that X is in the center of $\mathfrak{g}/\mathfrak{z}$. Since $\mathfrak{g}/\mathfrak{z}$ itself is readily seen to be centerless, we conclude that $X = 0$.

If $\big(g, [\,,]\big)$ is centerless then $[\,,]$ is a $\mathfrak{g}$-valued symplectic structure on $\mathfrak{g}$. Since the center is an abelian ideal, this includes every semisimple Lie algebra.

Example 6.4. For an infinite-dimensional example, suppose M is a smooth manifold of dimension at least 2. The wedge product $\omega : \Omega^1(M) \otimes \Omega^1(M) \to \Omega^2(M)$ is an $\Omega^2(M)$-valued symplectic structure on the vector space $\Omega^1(M)$. We will revisit this example in Chapter 9 as a first step towards analyzing the natural vector-valued symplectic structure on spaces of connections.

Definition 6.3. The *symplectic orthogonal* of a subspace $A \leq U$ is the subspace

$$A^\omega = \{v \in U \,|\, \omega(A, v) = 0\}$$

Example 6.5. Let $(U, \Pi_{\leq N}\omega_i)$ be the $\mathbb{R}^N$-valued symplectic vector space from Example 6.1. For $A \leq U$, we have

$$\begin{aligned}
u \leq A^{\oplus_i \omega_i} &\iff \forall u' \in U \,:\, (\oplus_i \omega_i)(u, u') = 0 \\
&\iff \forall u' \in U \,:\, \forall i \leq N \,:\, \omega_i(u, u') = 0 \\
&\iff \forall i \leq N \,:\, u \in A^{\omega_i}
\end{aligned}$$

We conclude that

$$A^{\oplus_i \omega_i} = \bigcap_i A^{\omega_i}$$

Example 6.6. Consider the $\mathbb{R}^3$-valued symplectic vector space $(\mathbb{R}^3, \times)$, where $\times$ denotes the cross product, and let e_1, e_2, e_3 be the standard coordinate basis vectors. Then

$$\langle e_1 \rangle^\omega = \{v \in \mathbb{R}^3 \,|\, v \times e_1 = 0\} = \langle e_1 \rangle$$

and

$$\langle e_1, e_2 \rangle^\omega = \{v \in \mathbb{R}^3 \,|\, v \times e_1 = v \times e_2 = 0\} = 0$$

Thus we have

A	A^ω
0	$\mathbb{R}^3$
ℓ	ℓ
w	0
$\mathbb{R}^3$	0

for any 1- and 2-dimensional subspaces $\ell, w \leq \mathbb{R}^3$, respectively.

This example illustrates the fact that, in contrast to the real-valued case, it is not true in general that $A^{\omega\omega} = A$.

Proposition 6.1. *Let $\mathfrak{a}$ be a subspace of a centerless Lie algebra $(\mathfrak{g}, [\,,\,])$.*

(i) The symplectic orthogonal $\mathfrak{a}^\omega$ is a Lie subalgebra of $\mathfrak{g}$.

(ii) If $(\mathfrak{g}, [\,,])$ *is semisimple and* $\mathfrak{a} \le \mathfrak{g}$ *is an ideal, then* $\mathfrak{a}^\omega$ *is also an ideal of* $\mathfrak{g}$ *and* $\mathfrak{a}^{\omega\omega} = \mathfrak{a}$

Proof. (i) For any $Y, Z \in \mathfrak{a}^\omega$ and any $X \in \mathfrak{a}$, we have

$$[X, [Y, Z]] = -[Z, [X, Y]] - [Y, [Z, X]] = 0$$

Thus $[Y, Z] \in \mathfrak{a}^\omega$ and, consequently, $\mathfrak{a}^\omega$ is a Lie subalgebra.

(ii) Recall that if $\mathfrak{g}$ is semisimple, then there is a decomposition $\mathfrak{g} = \bigoplus_{j \le N} I_j$ where each $I_j \le \mathfrak{g}$ is a simple ideal of $\mathfrak{g}$. As ideals are preserved under intersection, $\mathfrak{a} \cap I_j \le I_j$ is an ideal of $\mathfrak{g}$. Since I_j is simple, we have $\mathfrak{a} \cap I_j = I_j$ or 0. Thus, $\mathfrak{a}$ is of the form $\bigoplus_{j \in P} I_j$ for some $P \subseteq \{1, \ldots, N\}$. It follows that

$$\mathfrak{a}^\omega = \left(\bigoplus_{j \in P} I_j\right)^\omega = \bigoplus_{j \in P'} I_j$$

where $P' = \{1, \ldots, N\} \backslash P$. Applying this procedure twice yields $\mathfrak{a}^{\omega\omega} = \mathfrak{a}$.

□

It follows from Proposition 6.1 and Lemma 6.1, below, that if $\mathfrak{a} \le \mathfrak{g}$ is *not* a Lie subalgebra, then the containment $\mathfrak{a} \le \mathfrak{a}^{\omega\omega}$ is proper.

Let us now collect various properties on the symplectic orthogonal in vector-valued symplectic vector spaces.

Lemma 6.1. *For a V-valued vector space (U, ω) and subspaces $A, A_i, B, B_i \leq U$ $(i \leq N)$, we have*

(i) $U^\omega = 0$ *and* $0^\omega = U$.

(ii) *If* $A \leq B$, *then* $A^\omega \geq B^\omega$.

(iii) $A \leq A^{\omega\omega}$.

(iv) $A^\omega = A^{\omega\omega\omega}$.

(v) $\bigcap_i A_i^\omega = \big(\sum_i A_i\big)^\omega$.

(vi) $\sum_i A_i^\omega \leq \big(\bigcap_i A_i\big)^\omega$.

(vii) ω *descends to a symplectic structure on the quotient* $A/(A^\omega \cap A)$.

Proof. (i)–(iii) are straightforward.

(iv) Apply (ii) to (iii) to obtain $A^\omega \geq (A^{\omega\omega})^\omega$ and note that (iii) alone yields $A^\omega \leq (A^\omega)^{\omega\omega}$.

(v) A direct computation yields

$$\begin{aligned} u \in \big(\textstyle\sum_i A_i\big)^\omega &\iff \forall u \in \big(\textstyle\sum_i A_i\big) \,:\, \omega(u, u') = 0 \\ &\iff \forall i \leq N \,:\, \forall u_i \in A_i \,:\, \omega(u, u_i) = 0 \\ &\iff \forall i \leq N \,:\, u \in A^\omega \\ &\iff u \in \textstyle\bigcap_i A_i^\omega \end{aligned}$$

(vi) Applying (ii) to the inclusion $\bigcap_i A_i \leq A_j$, we deduce that $A_j^\omega \leq \left(\bigcap_i A_i\right)^\omega$ for all $j \leq N$. We conclude that $\sum_j A_j^\omega \leq \left(\bigcap_i A_i\right)^\omega$.

(vii) Let $u, u' \in A$, and observe that

$$\begin{aligned}\omega(u + C, u' + C) &= \omega(u, u') + \omega(u, C) + \omega(C, u') + \omega(C, C) \\ &= \omega(u, u')\end{aligned}$$

implies that ω descends to a well-defined form on $A/(A^\omega \cap A)$. Moreover, if $\omega(u, A) = 0$ then $u \in A^\omega$ and so $u \in A^\omega \cap A$. Thus the reduced form on $A/(A \cap A^\omega)$ is nondegenerate.

$\square$

Definition 6.4. Let V be a vector space and suppose that $\omega \in \Omega^2(V)$. For each $\alpha \in V^*$, the real-valued 2-form $\alpha \circ \omega \in \Lambda^2 V^*$ is called the *α-component* of ω.

Example 6.7. Consider again the space $(U, \oplus_i \omega_i)_{i \leq N}$ of Example 6.1. Denote by $\{e_i\}_{i \leq N}$ the standard basis of $\mathbb{R}^N$ and let $\{\alpha_i\}_{i \leq N}$ be the dual basis. Since

$$(\alpha_j \oplus_i \omega_i)(X, Y) = \alpha_j(\omega_i(X, Y))_i = \omega_j(X, Y)$$

we have that the α_i-component of $\oplus_i \omega_i$ is the ith component, ω_i. For general $\alpha \in$

$(\mathbb{R}^N)^*$ the α-component of ω is a weighted sum of the forms $(\omega_i)_i$,

$$\omega_\alpha = \sum_i c_i \omega_i$$

where the coefficients are given by $c_i = \alpha(e_i)$.

Example 6.8. Fix a vector $v \in \mathbb{R}^3$. Under the standard identification of $\mathbb{R}^3$ with its dual, the v-component of the cross product $\times$ on $\mathbb{R}^3$ is given by the so-called *triple product*,

$$v \cdot (u_1 \times u_2)$$

where $\cdot$ denotes the dot product on $\mathbb{R}^3$. Equivalently, the v-component of $\times$ evaluated at $(u_1, u_2) \in (\mathbb{R}^3)^2$ is the volume of the parallelepiped with side edges v, u_1, u_2.

Recall that the *rank* of a bilinear form $\omega : U^{\otimes 2} \to \mathbb{R}$ is its rank as a map $U \to U^*$. Equivalently, the rank of ω is the dimension of any subspace $\hat{U} \leq U$ which is maximal subject to the condition that $\omega|_{\hat{U}}$ is nondegenerate.

Example 6.9. Let ω be a $\mathbb{R}$-valued symplectic structure on the vector space U and consider the $\mathbb{R}^2$-valued symplectic vector space $(U, \omega \oplus 0)$. For $v \in \mathbb{R}^2$, we have

$$\operatorname{rank}(v \cdot \omega \oplus 0) = \begin{cases} 0 & \text{if } v \in 0 \oplus \mathbb{R} \\ \dim U & \text{otherwise} \end{cases}$$

Observe that the generic component of $\omega \oplus 0$ has maximal rank. As the following result establishes, this property is true in general.

Proposition 6.2. *Let (U, ω) be a V-valued symplectic vector space. The typical component of ω has maximal rank.*

Proof. Let $\alpha \in V^*$. Then $\alpha\omega$ is nondegenerate on some k-dimensional subspace $U \subseteq T$, where k is the rank of α. Fix a basis $(e_i)_i$ of U and let $(c_{ij})_{ij}$ be the corresponding V-valued matrix representation of ω on U. Since the assignment $\beta \mapsto \big(\beta(c_{ij})\big)_{ij}$ is linear, it follows that $\phi : V^* \to \mathbb{R}$, $\beta \mapsto \det\big(\beta(c_{ij})\big)_{ij}$ is analytic. As $\phi(\alpha) \neq 0$, it follows that ϕ is vanishes on a set of positive codimension. Since $\beta\omega$ is degenerate on U precisely when $\phi(\beta) = 0$, we conclude that the generic $\beta\omega$ is nondegenerate on U and thus has rank at least k. □

The rank of the typical component of $(U, \oplus_i \omega_i)$ is $\dim U$, and the rank of *every* nonzero component of $(\mathbb{R}^3, \times)$ is 2.

Recall the classification scheme of subspaces $A \leq U$ from Definition 3.3.

term	condition
isotropic	$A \leq A^\omega$
coisotropic	$A^\omega \leq A$
Lagrangian	$A^\omega = A$
symplectic	$A^\omega \cap A = \varnothing$

We will adapt this terminology to the theory of vector-valued symplectic vector spaces without alteration, save for an implicit reading of "symplectic" as "V-valued symplectic". Due to the fact that $A^{\omega\omega} \leq A$ is not in general an equality, there is no longer a symmetry between isotropic and coisotropic subspaces. If $A \leq U$ is coisotropic, that is, that $A \leq A^\omega$, then Lemma 6.1 implies that $A^\omega \geq A^{\omega\omega}$, that is, that A^ω is isotropic. The reverse implication obtains precisely when $A = A^{\omega\omega}$.

Thus, in contrast to the classical case, where the involution $A \mapsto A^\omega$ yields a bijection between isotropic and coisotropic subspaces, a vector-valued symplectic vector space (U, ω) has only at least as many isotropic subspaces as coisotropic subspaces. Every isotropic subspace A yields a coisotropic subspace A^ω but not necessarily vice versa.

6.2 Quotients and Refinements

Definition 6.5. Let (U, ω) and (U', ω') be V and V'-symplectic vector spaces, respectively. A *weak morphism of vector-valued symplectic vector spaces*,

$$f : (U, \omega) \to (U', \omega')$$

is a pair of linear maps

$$f_0 : U \to U'$$

$$f_1 : V \to V'$$

such that $f_0^*\omega' = f_1 \circ \omega$. We call f an *automorphism* when $V = V'$ and $f_1 = 1_V$.

Note that if $q : (U, V) \to (U, V')$ is a morphism with q_0 and q_1 surjective, then for any W-valued vector space (T, ν) and any morphism $f : U \to T$, there is a unique morphism $\bar{f} : U' \to T$ such that the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "q"'] & T \\ U' \arrow[ur, dashed, "\exists! \bar{f}"'] & \end{tikzcd}$$

This motivates the following definition.

Definition 6.6. A morphism $q : (U, \omega) \to (U', \omega')$ of vector-valued symplectic vector spaces is said to be a *quotient map* if q_0 and q_1 are surjective. If additionally $U' \leq U$, $V' \leq V$, and q_0 and q_1 extend the identity on U' and V', respectively, then (U', ω') is called a *quotient* of (U, ω). If $U = U'$, we say that q is a *reduction of coefficients*.

The dual notion is that of a *refinement*. We say that (U', ω') *refines* (U, ω) if there is an injection $i : U \to U'$ with $\omega' = i^*\omega$.

Example 6.10. Each (U, ω_i) is obtained by reducing the coefficients of $(U, \oplus_i \omega_i)$ from $\mathbb{R}^N$ to $\mathbb{R}$. Equivalently, $(U, \oplus_i \omega_i)$ refines each (U, ω_i).

6.3 Reduction of V-Symplectic Vector Spaces

Lemma 6.2. *Let V be a vector space, let (U, ω) be a V-valued symplectic vector space, and let $A \leq U$ be an isotropic subspace. Then*

$$A \leq A^{\omega\omega} \leq A^{\omega}$$

and ω descends to a well-defined V-valued form ω_0 on A^{ω}/A. Moreover, ω_0 is a V-valued symplectic structure on A^{ω}/A precisely when $A^{\omega\omega} = A$.

Proof. Since $A \leq A^{\omega}$, Lemma 6.1, part (ii), yields $A^{\omega\omega} \leq A^{\omega}$. Lemma 6.1, part (iii), implies that $A \leq A^{\omega\omega}$. Thus we obtain the chain of inclusions

$$A \leq A^{\omega\omega} \leq A^{\omega}$$

Consequently, for $u, u' \in A^{\omega}$, the fact that A is contained in A^{ω} yields

$$\begin{aligned}\omega(u + A, u' + A) &= \omega(u, u') + \omega(u, A) + \omega(A, u') + \omega(A, A)\\ &= \omega(u, u')\end{aligned}$$

which in turn implies that ω descends to a well-defined form ω_0 on A^{ω}/A. The kernel of ω_0 consists of precisely those elements $u + A \in A^{\omega}/A$ for which

$$0 = \omega(u + A, A^{\omega}) = \omega(u, A^{\omega})$$

that is, $u \in A^{\omega\omega}$. Thus $\ker \omega = A^{\omega\omega}/A$, which is trivial precisely when $A = A^{\omega\omega}$. $\square$

We call $(A^{\omega}/A, \omega_0)$ the *reduced vector space* for the isotropic space A, and ω_0 is said to be the *reduced form* or *reduced structure* on A^{ω}/A. We emphasize that, unlike in the classical case, ω_0 is not guaranteed to be nondegenerate

Chapter 7

The V-Hamiltonian Formalism

Having developed the basic theory of vector-valued symplectic vector spaces, we turn our attention to the global setting. This chapter parallels much of the exposition in the Chapter 3, beginning with the definition of a vector-valued symplectic structure and culminating with a symplectic reduction theorem.

7.1 V-Symplectic Manifolds

We begin with the fundamental definition of vector-valued symplectic geometry.

Definition 7.1. Let V be a vector space. A *V-valued symplectic manifold*, or *V-symplectic manifold* is a smooth manifold M equipped with a closed 2-form $\omega \in \Omega^2(M, V)$, called a *$V$-valued symplectic structure* or a *V-symplectic structure*, which is nondegenerate in the sense that $\iota_X \omega \neq 0$ for every $X \in \mathfrak{X}(M)$.

When we wish to emphasize the distinction, we will refer to a symplectic structure in the usual sense as a *classical symplectic structure*. When the difference between the two is clear or irrelevant, we refer to either construction as a *symplectic structure*.

As for V-symplectic vector spaces, we will consider three primary examples.

1. The direct sum $\oplus_i \omega_i \in \Omega^1(M, \mathbb{R}^N)$ of a collection of symplectic forms $\{\omega_i\}_{i \leq N}$ on M.

2. A compact connected semisimple Lie group G, equipped with the $\mathfrak{g}$-valued 2-form $-\mathrm{d}\theta$, where $\theta \in \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form on G.

3. The space $\mathrm{Hom}(TQ, V)$ equipped with the V-valued 2-form $-\mathrm{d}\theta$, where $\theta \in \Omega^1\big(\mathrm{Hom}(TQ, V), V\big)$ is the fundamental 1-form on $\mathrm{Hom}(TQ, V)$.

We will explore the first two in this exposition; the third is treated extensively in Chapter 8.

It is interesting to compare this list to Kirillov's [41] three sources of finite-dimensional classical symplectic manifolds,

1. algebraic submanifolds of the complex projective space $\mathbb{C}P^N$.

2. the coadjoint orbits of a compact semisimple Lie group,

3. the phase space $(T^*Q, -\mathrm{d}\theta)$ of a smooth manifold Q.

We will see that the coadjoint orbits are obtained from the Hamiltonian reduction of $(G, -\mathrm{d}\theta)$ by a suitable action of a maximal torus $T \leq G$. The classical phase space

$(T^*Q, -\mathrm{d}\theta)$ is simply the $\mathbb{R}$-valued instance of $\big(\mathrm{Hom}(TQ, V), -\mathrm{d}\theta\big)$. By contrast, there is no apparent candidate for the vector-valued generalization of the projective space $\mathbb{C}P^N$. This stems from the fact that $\mathbb{C}P^N$ is the reduction of $T^*\mathbb{R}^N$ by a circle action which does not preserve the base $\mathbb{R}^N$, whereas such actions on $\mathrm{Hom}(TQ, V)$ do not generally yield symplectic spaces under Hamiltonian reduction.

Example 7.1. Let M be a smooth even-dimensional manifold and suppose that $\{\omega_i\}_{i\leq N}$ is a collection of symplectic forms on M. Then the map $\oplus_i\omega_i \in \Omega^2(M, \mathbb{R}^N)$, given by

$$(\oplus_i\omega_i)(X, Y) = \oplus_i\big[\omega_i(X, Y)\big]$$

is evidently an $\mathbb{R}^N$-valued symplectic form on M. The components of ω consist of the linear combinations $\sum_i c_i\omega_i$ of the symplectic forms $\{\omega_i\}_i$.

Note that it is not necessary for each component $\omega_i \in \Omega^2(M)$ to be nondegenerate; it suffices that for each point $p \in M$ at least one component ω_i is nondegenerate. This condition is far from necessary, as the following example illustrates.

Example 7.2. Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds and let $\pi_i : M_1 \times M_2 \to M_i$ be the ith projection map. The form $\pi_i^*\omega_i \in \Omega^2(M_1 \times M_2)$ is nondegenerate on the subbundle $\pi_i^*TM_i \leq T(M_1 \times M_2)$ and zero on the complementary subbundle $\pi_j^*TM_j \leq T(M_1 \times M_2)$ $(i \neq j)$. Thus, while neither $\pi_1^*\omega_1$ nor $\pi_2^*\omega_2$ is a symplectic form on $M_1 \times M_2$, the product $\pi_1^*\omega^1 \times \pi_2^*\omega^2$ is a $\mathbb{R}^2$-valued symplectic form.

This example can be extended to the product of any number of factors (M_j, ω_j).

Let G be a Lie group and recall that the *Maurer-Cartan form* $\theta \in \Omega^1(G, \mathfrak{g})$ is defined by

$$\theta_g(X) = (\lambda_{g^{-1}})_* X \in \mathfrak{g}$$

where $g \in G$, $X \in T_g G$, and $\lambda : G \curvearrowright G$ is the left regular representation. When G is compact and semisimple, the *Cartan* 3*-form* $\chi \in \Omega^3(G, \mathbb{R})$ is defined to be

$$\chi(X, Y, Z) = \frac{1}{12}\langle X, [\theta(Y), \theta(Z)]\rangle$$

where $\langle\,,\rangle$ denotes the Killing metric on G.

Example 7.3. Let G be a semisimple Lie group and let $\theta \in \Omega^1(M, \mathfrak{g})$ denote the Maurer-Cartan form on G. The Maurer-Cartan theorem asserts that $-\mathrm{d}\theta = \theta^*[\,,]$, which is nondegenerate by the semisimplicity of G. Thus $\omega = -\mathrm{d}\theta$ is a $\mathfrak{g}$-valued symplectic form on G. When G is compact and $Y \in \mathfrak{g}$, the Y-component of ω is given by

$$\langle Y, \omega\rangle = 12\, \iota_{\bar{Y}}\chi$$

where $\bar{Y} \in \mathfrak{X}(G)$ is the unique left-invariant extension of Y.

Observe that $-\mathrm{d}\theta$ is nondegenerate precisely when the Lie algebra $\mathfrak{g}$ is centerless. We impose the additional condition of semisimplicity to obtain stronger results on the reduced space later on. This example demonstrates that it is possible for a vector-valued symplectic form on a compact manifold to be exact, which constitutes a further difference with the real-valued case. In fact, as it is a consequence of Theorem 21.2 of [11] that a compact semisimple Lie group G never admits a symplectic structure, this example is particularly representative of the difference of behavior between the classical and vector-valued cases.

There are two natural generalizations of the classical notion of symplectomorphism.

Definition 7.2. Let (M, ω) and (M', ω') be V and V'-symplectic manifolds, respectively. A *weak symplectomorphism*

$$f : (M, \omega) \to (M', \omega')$$

consists of a smooth map $f_0 : M \to M'$ and a linear map $f_1 : V \to V'$ such that $f_0^* \omega' = f_1 \circ \omega$. We say that f is a *symplectomorphism* when $V = V'$ and $f_1 = 1_V$.

We will be particularly interested in the case where $M' = M$.

Example 7.4. The map $f : M \to M$ is a symplectic transformation of $(M, \oplus_i \omega_i)$ if and only if f is symplectic with respect to each ω_i, that is, $f^* \omega_i = \omega_i$.

Example 7.5. Fix $g \in G$ and let $\lambda_g : G \to G$ be the transformation given by left-multiplication. For any $X \in T_h G$, we have

$$\lambda_g^* \theta_{gh}(X) = (\lambda_{(gh)^{-1}})_* (\lambda_g)_* X = \theta_h(X)$$

from which $\lambda_g^*\theta = \theta$. We conclude that left-multiplication is symplectic with respect to $-\mathrm{d}\theta$.

Remark 7.1. There is an analogue of the classical symplectic volume on a $2n$-dimensional V-valued manifold (M, ω),

$$\operatorname{vol} M = \frac{1}{n!}\int_M \omega^n \in V^{\otimes n}$$

The designation is purely formal; $\operatorname{vol} M$ does not in general behave like a volume. For example, considering $(G, -\mathrm{d}\theta)$, we have

$$\operatorname{vol} G = \frac{1}{n}\int_G (-\mathrm{d}\theta)^n = 0$$

where $n = \dim G/2$. More generally, for an even V-symplectic manifold (M^{2n}, ω), we see that $\operatorname{vol} M = 0$ if and only if $\omega^n \in \Omega^{2n}(M, V^{\otimes n})$ is exact. The symplectic volume of an odd manifold vanishes by default.

7.2 Special Submanifolds

Definition 7.3. Let (M, ω) be a V-valued symplectic manifold, and let $N \subseteq M$ be a submanifold. Then N is said to be *Lagrangian* (resp. *isotropic*, *coisotropic*, *symplectic*) when the subspace $T_xN \leq T_xM$ is Lagrangian (resp. isotropic, coisotropic, symplectic) for every $x \in N$.

Proposition 7.1. *Suppose that (M, ω) is a V-valued symplectic manifold and that $\omega' \in \Omega^2(M, V')$ refines $\omega \in \Omega^2(M, V)$. Then if $N \subseteq M$ is symplectic (resp. coisotropic) with respect to ω, then N is symplectic (resp. coisotropic) with respect to ω'.*

Proof. Fix $x \in N$. Since ω' is a refinement of ω, it follows that

$$T_x N^{\omega'} \leq T_x N^{\omega}$$

Thus, if N is symplectic with respect to ω, then

$$T_x N \cap T_x N^{\omega'} \leq T_x N \cap T_x N^{\omega} = 0$$

and N is symplectic with respect to ω'. If N is coisotropic with respect to ω, then

$$T_x N^{\omega'} \leq T_x N^{\omega} T_x N$$

and N is coisotropic with respect to ω. ☐

Proposition 7.2. *If L is a Lagrangian submanifold of a V-valued symplectic manifold (M, ω), then L is not contained in any Lagrangian manifold of strictly greater dimension.*

Proof. Suppose that $\bar{L} \subseteq M$ is a Lagrangian submanifold with $L \subseteq \bar{L}$ and let $x \in L$.

Since L and $\bar{L}$ are Lagrangian, and $T_xL \leq T_x\bar{L}$, Lemma 6.1 implies that

$$T_xL = T_xL^\omega \geq T_x\bar{L}^\omega = T_x\bar{L}$$

We conclude that $T_xL = T_x\bar{L}$ and thus $\dim L = \dim \bar{L}$. $\square$

7.3 Symplectic and Hamiltonian Actions

We continue the analogy with the classical situation with the introduction of V-symplectic actions, Hamiltonian actions, comoment maps, and moment maps.

Definition 7.4. Let (M, ω) be a V-symplectic manifold. The action $\lambda : G \curvearrowright M$ is called a *symplectic action* when $\lambda : G \to \mathrm{Sym}(M, \omega)$, where $\mathrm{Sym}(M, \omega) \leq \mathrm{Diff}(M)$ is the group of symplectic transformations of M. The vector field $X \in \mathfrak{X}(M)$ is said to be a *symplectic vector field* when X is tangent to a symplectic action. Equivalently, X is symplectic when $\mathcal{L}_X\omega = 0$.

Definition 7.5. Let (M, ω) be a V-symplectic manifold and suppose that $f \in C^\infty(M, V)$ and $X \in \mathfrak{X}(M)$ satisfy

$$-\iota_X\omega = \mathrm{d}f$$

Then X is called the *Hamiltonian vector field* of f, and f is called the *Hamiltonian function* of X. We also call X the *symplectic gradient* of f and denote it by $s\text{-grad}\, f$.

More generally, f (resp. X) is said to be *Hamiltonian* if it possesses a Hamiltonian vector field (resp. Hamiltonian function).

We summarize the primary vector-valued constructions beside their classical counterparts in the table below.

	classical	V-valued	
symplectic form	$\omega \in \Omega^2(M, \mathbb{R})$	$\omega \in \Omega^2(M, V)$	
Hamiltonian v.f.	$f \in C^\infty(M)$	$f \in C^\infty(M, V)$	$\omega(\,\cdot, X_f) = df$
comoment map	$\tilde{\mu} : \mathfrak{g} \to C^\infty(M)$	$\tilde{\mu} : \mathfrak{g} \to C^\infty(M, V)$	
moment map	$\mu : M \to \mathfrak{g}^*$	$\mu : M \to \mathrm{Hom}(\mathfrak{g}, V)$	

Proposition 7.3. *Let (M, ω) be a V-symplectic manifold and let $X \in \mathfrak{X}(M)$ be a Hamiltonian vector field with Hamiltonian function $f \in C^\infty(M)$.*

(i) The vector field X is symplectic.

(ii) The flow of X preserves f.

Proof. This follows as

(i) $\mathcal{L}_X \omega = \mathrm{d}\iota_X \omega = -\mathrm{dd}f = 0$, and

(ii) $Xf = \mathrm{d}f(X) = \omega(X, X) = 0$.

∎

In contrast to the real-valued case, it is not in general true that every function $f \in C^\infty(M, V)$ is Hamiltonian. This is apparent by observing that the assignment

$$\iota\,\omega_x : T_xM \to T_x^*M \otimes V, \qquad x \in M$$
$$X \mapsto \iota_X\omega_x$$

cannot be an isomorphism when $\dim V \geq 2$. The function $f \in C^\infty(M, V)$ is Hamiltonian precisely when $\mathrm{d}f_x$ lies in the image of $\iota\,\omega_x$ for every $x \in M$. The vector field $X \in \mathfrak{X}(M)$ is Hamiltonian when $(\iota\,\omega)^{-1}X \in \Omega^1(M, V)$ is exact.

We will denote by $C^\infty_H(M)$ the vector space of Hamiltonian functions.

Definition 7.6. We define the *Poisson bracket* $\{\,,\}$ on the space of Hamiltonian functions $C^\infty_H(M)$ by

$$\{f, f'\} = -\omega(X_f, X_{f'})$$

Just as in the classical situation, the Poisson bracket is a Lie bracket on $C^\infty_H(M, V)$ and the symplectic gradient s-grad : $C^\infty_H(M, V) \to \mathfrak{X}(M)$ is a Lie algebra anti-homomorphism.

Example 7.6. The function $f = (f_i)_i \in C^\infty(M, \mathbb{R}^N)$ is Hamiltonian on $(M, \oplus_i\omega_i)$, with Hamiltonian vector field $X \in \mathfrak{X}(M)$, if and only if X is the Hamiltonian vector field of f_i with respect to ω_i for each $i \leq N$.

Example 7.7. Consider $(G, -\mathrm{d}\theta)$, where $\theta \in \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form.

Fix $Y \in \mathfrak{g}$ and let $\bar{Y} \in \mathfrak{X}(G)$ be the *right* invariant extension of Y. Since the integral flow of $\bar{Y}$ corresponds to left multiplication by $\exp(tY)$, and since we have shown left multiplication to preserve θ, it follows that $\mathcal{L}_{\bar{Y}}\theta = 0$. Therefore,

$$-\iota_{\bar{Y}}\omega = \iota_{\bar{Y}}\mathrm{d}\theta = -\mathrm{d}\theta(\bar{Y})$$

Now, for every $g \in G$,

$$\theta_g(\bar{Y}) = (\lambda_{g^{-1}}\rho_g)_* Y = \mathrm{Ad}_g^{-1} Y$$

It follows that the function

$$\begin{aligned} \mathrm{Ad}^{-1} Y : G &\longrightarrow \mathfrak{g} \\ g &\longmapsto \mathrm{Ad}_g^{-1} Y \end{aligned}$$

is Hamiltonian, with associated Hamiltonian vector field $-\bar{Y}$.

Definition 7.7. Suppose the Lie group G acts on the V-symplectic manifold (M, ω). A *weak comoment map* is any linear map

$$\tilde{\mu} : \mathfrak{g} \to C^\infty_H(M, V)$$

which lifts the fundamental vector fields of λ to the space of Hamiltonian functions $C^\infty_H(M, V)$, as indicated in the following diagram.

$$\begin{tikzcd}
 & C^\infty_H(M,V) \arrow[d, "s\text{-grad}"] \\
\mathfrak{g} \arrow[ur, dashed, "\tilde{\mu}"] \arrow[r, "\lambda_*"'] & \mathfrak{X}(M)
\end{tikzcd}$$

If $\tilde{\mu}$ is a morphism of Lie algebras, then it is called a *comoment map*. The *(weak) moment map* associated to a (weak) comoment map $\tilde{\mu}$ is the assignment

$$\mu : M \to \mathrm{Hom}(\mathfrak{g}, V)$$

given by

$$\mu(x)(Y) = \tilde{\mu}(Y)(x)$$

for $x \in M$ and $Y \in \mathfrak{g}$. When the action of G admits a moment map μ, the action is said to be *Hamiltonian* and the quadruple (M, ω, G, μ) is a *V-valued Hamiltonian system* or a *V-Hamiltonian system.*

Definition 7.8. The *coadjoint action* $\mathrm{Ad}^* : G \curvearrowright \mathrm{Hom}(\mathfrak{g}, V)$ is given by

$$(\mathrm{Ad}^*_g \alpha)(Y) = \alpha(\mathrm{Ad}_g^{-1} Y)$$

for $g \in G$, $\alpha \in \mathrm{Hom}(\mathfrak{g}, V)$, and $X \in \mathfrak{g}$.

Lemma 7.1. *Let (M, ω, G, μ) be a V-Hamiltonian system.*

(i) If $\alpha \in \mathrm{Hom}(\mathfrak{g}, V)$*, then the assignment*

$$\mu + \alpha : M \to \mathrm{Hom}(\mathfrak{g}, V)$$
$$x \to \mu(x) + \alpha$$

is a weak moment map. In particular, the set of weak moment maps compatible with (M, ω, G) *is a* $\mathrm{Hom}(\mathfrak{g}, V)$*-affine space. If additionally* α *vanishes on commutators* $[Y, Z] \in \mathfrak{g}$ *(*$Y, Z \in \mathfrak{g}$*) then* α *is a moment map. Consequently, the set of moment maps is a* $[\mathfrak{g}, \mathfrak{g}]^0$*-affine space, where* $[\mathfrak{g}, \mathfrak{g}]^0$ *denotes the annihilator of* $[\mathfrak{g}, \mathfrak{g}] \leq \mathfrak{g}$ *in* $\mathrm{Hom}(\mathfrak{g}, V)$*.*

(ii) The assignment $\mu' : M \to \mathrm{Hom}(\mathfrak{g}, V)$ *is a moment map for* (M, ω, G) *precisely when*

$$\langle \mu_* X, Y \rangle = \omega(X, \underline{Y}_x)$$

for all $x \in M$*,* $X \in T_x M$*, and* $Y \in \mathfrak{g}$*. The map* $\langle \, , \rangle : \mathrm{Hom}(\mathfrak{g}, V) \otimes \mathfrak{g} \to V$ *denotes the natural pairing.*

(iii) If G *is connected, then the moment map* μ *intertwines the actions of* $G \curvearrowright M$ *and* $\mathrm{Ad}^* : G \curvearrowright \mathrm{Hom}(\mathfrak{g}, V)$*.*

(iv) Let $H \leq G$ be a Lie subgroup and define the restricted moment map

$$\mu|_{\mathfrak{h}} : M \to \operatorname{Hom}(\mathfrak{h}, V)$$

$$x \mapsto \mu(x)|_{\mathfrak{h}}$$

Then $(M, \omega, H, \mu|_{\mathfrak{h}})$ is a V-Hamiltonian system.

The proofs of these statements are so similar to their classical counterparts that it would be redundant to present them here. We refer instead to the corresponding proofs in Chapter 3.

Example 7.8. The action of G on $(M, \oplus_i \omega_i)$ is Hamiltonian if and only if it is Hamiltonian with respect to each ω_i ($i \leq N$). In this case, a moment map is given by $\oplus_i \omega_i : M \to (\mathfrak{g}^*)^N \cong \operatorname{Hom}(\mathfrak{g}, \mathbb{R}^N)$, where $\mu_i : M \to \mathfrak{g}^*$ is a moment map for the action of G on (M, ω_i).

Proposition 7.4. *The left regular action of G on $(G, -\mathrm{d}\theta)$ is Hamiltonian with moment map $-\operatorname{Ad}^{-1} : G \to \operatorname{Aut}\mathfrak{g}$.*

Proof. Let $Y \in \mathfrak{g}$ and let $\bar{Y}$ be the right invariant extension of Y to $\mathfrak{X}(G)$. We have already shown that $-\operatorname{Ad}^{-1} Y : G \to \mathfrak{g}$ is a Hamiltonian function for $\bar{Y}$. Since $\bar{Y}$ is also the induced vector field $\lambda_* Y$ for the left regular action λ, it follows that $Y \mapsto -\operatorname{Ad}^{-1} Y$ is a weak moment map for λ. For $Y, Z \in \mathfrak{g}$, we have

$$\{-\operatorname{Ad}^{-1} Y, -\operatorname{Ad}^{-1} Z\} = -[\theta\bar{Y}, \theta\bar{Z}] = -[\operatorname{Ad} Y, \operatorname{Ad} Z] = -\operatorname{Ad}[Y, Z]$$

and thus $-\mathrm{Ad}^{-1}$ is a morphism of Lie algebras. We conclude that Ad is a moment map for λ. $\square$

7.4 Vector-Valued Symplectic Reduction

Theorem 7.1 (Vector-valued symplectic reduction)**.** *Let (M, ω, G, μ) be a V-valued Hamiltonian system. If G is connected and the* reduced space $M_0 = \mu^{-1}(0)/G$ *is smooth, then there is a unique V-valued 2-form $\omega_0 \in \Omega^2(M_0, V)$ such that*

$$\pi^*\omega_0 = i^*\omega$$

for the inclusion $i : \mu^{-1}(0) \hookrightarrow M$ and projection $\pi : \mu^{-1}(0) \to M_0$. Moreover, ω_0 is closed and the kernel of ω_0 at $\pi x \in M_0$ is equal to

$$\ker_x \omega_0 = \pi_*\big(T_x\mu^{-1}(0)^\omega/\underline{\mathfrak{g}}_x\big) \leq T_{\pi x}M_0$$

Proof. Fix $x \in \mu^{-1}(0)$. Since Lemma 7.1 implies that

$$\omega(X, \underline{Y}_x) = \langle \mu_* X, Y\rangle$$

for all $X \in T_xM$ and $Y \in \mathfrak{g}$, we have

$$\underline{\mathfrak{g}}_x^\omega = \mathrm{d}\mu_x^{-1}(0) = T_x\mu^{-1}(0)$$

Lemma 7.1 also asserts that $\mu : G \to \mathrm{Hom}(\mathfrak{g}, V)$ is equivariant, so that

$$\mu(G \cdot x) = \mathrm{Ad}^*_G\, 0 = 0$$

and thus $G \cdot x \subseteq \mu^{-1}(0)$ and $\mathfrak{g}_x \leq T\mu^{-1}(0)$. Consequently, $\underline{\mathfrak{g}}_x^\omega \leq T_x\mu^{-1}(0)$, and thus Lemma 6.2 ensures that ω_x descends to a unique V-valued symplectic structure $\omega_{0,x}$ on the vector space $T_{\pi x}(\mu^{-1}(0)/G) \cong T_x\mu^{-1}(0)/\underline{\mathfrak{g}}_x$. Thus we have shown that $i^*\omega$ descends to a unique 2-form $\omega_0 \in \Omega^2(M, V)$.

Taking the exterior derivative of both sides of the equality $\pi^*\omega_0 = i^*\omega$ yields

$$\pi^*\mathrm{d}\omega_0 = i^*\mathrm{d}\omega = 0$$

Thus, as $\pi : \mu^{-1}(0) \to M_0$ is a submersion, we conclude that $\mathrm{d}\omega_0 = 0$ and $\ker \omega_0 \leq TM_0$ is a distribution whenever it has constant rank. Since the fiberwise description of the kernel of ω_0 at $\pi x \in M_0$ is an immediate consequence of Lemma 6.2, this completes the proof. $\square$

Theorem 7.2 (Vector-Valued Stabilizer Reduction)**.** *Let (M, ω, G, μ) be a V-Hamiltonian system, fix $\alpha \in \mathrm{Hom}(\mathfrak{g}, V)$, and let G_α be the* Ad^**-stabilizer of α. Put $\mu_\alpha = \mu|_{\mathfrak{g}_\alpha} : M \to \mathrm{Hom}(\mathfrak{g}, V)$. If the reduced space $M_\alpha = \mu_\alpha^{-1}(\alpha)/G_\alpha$ is smooth, then there is a unique V-valued 2-form $\omega_\alpha \in \Omega^2(M_\alpha, V)$ such that*

$$\pi^*\omega_\alpha = i^*\omega$$

where $i : \mu|_{\mathfrak{g}_\alpha}^{-1}(\alpha) \hookrightarrow M$ *is the inclusion and* $\pi : \mu|_{\mathfrak{g}_\alpha}^{-1}(\alpha) \to M_\alpha$ *is the projection. Moreover,* $\omega_\alpha \in \Omega^2(M, V)$ *is closed.*

Proof. Lemma 7.1 implies that $(M, \omega, G_\alpha, \mu|_{\mathfrak{g}_\alpha})$ is a V-Hamiltonian system. For any $Y, Z \in \mathfrak{g}_\alpha$ we have

$$\alpha\,[Y, Z] = (\mathrm{ad}_Z^*\alpha)(Y) = 0$$

Thus $\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha]^0$ and a second application of Lemma 7.1 implies that $\mu|_{\mathfrak{g}_\alpha} - \alpha$ is a moment map for the action of G. The result follows by Theorem 7.1, since

$$\mu|_{\mathfrak{g}_\alpha}^{-1}(\alpha) = (\mu|_{\mathfrak{g}_\alpha} - \alpha)^{-1}(0)/G_\alpha$$

$\square$

Example 7.9. The underlying space of the reduction of the Hamiltonian system $(M, \oplus_i \omega_i, G, \oplus_i \mu_i)$ is equal to the intersection of the reduced spaces of each (M, ω_i, G, μ_i). That is,

$$M_0 = \Big(\bigcap_i \mu_i^{-1}(0)\Big)/G = \bigcap_i \big(\mu_i^{-1}(0)/G\big)$$

Example 7.10. Since the adjoint action is always an automorphism, the reduction of $(G, -d\theta, G, -\mathrm{Ad}^{-1})$ at $0 \in \operatorname{Aut}\mathfrak{g}$ is empty. Thus let us determine the reduction at the identity transformation $1 \in \operatorname{Aut}\mathfrak{g}$. For $g \in G$, we have

$$\mathrm{Ad}_g^* 1 = 1 \iff \mathrm{Ad}_g = 1 \iff g \in Z(G)$$

and hence the stabilizer G_1 of $1 \in \operatorname{Aut}\mathfrak{g}$ is equal to the center $Z(G)$. Since G is semisimple, it follows that Z is discrete and thus the Lie algebra $\mathfrak{z} = 0_{\mathfrak{g}}$ is trivial. We conclude that

$$\mu|_{\mathfrak{g}_1}^{-1}(1_{\operatorname{Aut}\mathfrak{g}}) = -\mathrm{Ad}^{-1}|_{0_{\mathfrak{g}}}(1) = G$$

and so the reduced space at $1 \in \operatorname{Aut}\mathfrak{g}$ is the adjoint group

$$\mu|_{\mathfrak{g}_1}^{-1}(1)/G_1 = G/Z$$

The reduced form is equal to $-\mathrm{d}\bar{\theta}$, where $\theta \in \Omega^1(G/Z, \mathfrak{g})$ is the Maurer-Cartan form on G/Z. In contrast to the previous example, the reduced space is symplectic.

Example 7.11. Now suppose that $T \leq G$ is a maximal torus, with Lie algebra $\mathfrak{t} \leq \mathfrak{g}$, and consider the $\mathfrak{g}$-Hamiltonian system $(G, -\mathrm{d}\theta, T, -\mathrm{Ad}^{-1}|_{\mathfrak{t}})$. Once again, the 0-level set

$$\mu^{-1}(0) = -\mathrm{Ad}|_{\mathfrak{t}}^{-1}\, 0_{\mathrm{Hom}(\mathfrak{t},\mathfrak{g})}$$

is empty, as $\mathrm{Ad}_g \in \operatorname{Aut}\mathfrak{g}$ is an automorphism for all $g \in G$. Let us determine the reduction at the negativ inclusion $-i \in \mathrm{Hom}(\mathfrak{t}, \mathfrak{g})$. For every $t \in T$, the adjoint transformation $\mathrm{Ad}_t \in \operatorname{Aut}\mathfrak{t}$ acts trivially on $\mathfrak{t}$. Consequently, the stabilizer $T_{-i} \leq T$

of $-i$ under the action $\mathrm{Ad}^* : T \curvearrowright \mathrm{Hom}(\mathfrak{t}, \mathfrak{g})$ is all of T. Now,

$$\mu(g) = -i \iff \mathrm{Ad}_g^{-1}|_{\mathfrak{t}} = 1_{\mathrm{Aut}\,\mathfrak{t}} \iff g \in C_G(T)$$

where $C_G(T)$ is the centralizer of T. Since T is a maximal torus, it follows that $C_G(T) = T$ and thus $\mu^{-1}(0) = T$. Therefore, the reduced space

$$\mu|_{\mathfrak{g}_i}^{-1}(i)/G_i = T/T = 1$$

is a point.

Example 7.12. Consider $\mathbb{R}^3$ with the $\mathbb{R}^3$-valued symplectic form $\omega \in \Omega^2(\mathbb{R}^3, \mathbb{R}^3)$ defined by

$$\omega(X_p, Y_p) = (X \times Y)_p$$

for $p \in \mathbb{R}^3$ and $X_p, Y_p \in T_p\mathbb{R}^3$. The action of $\mathbb{R}$ by vertical translations is Hamiltonian with respect to ω and a moment map for this action is

$$\begin{aligned} \mu : \mathbb{R}^3 &\to \mathrm{Hom}(\mathbb{R}, \mathbb{R}^3) \\ p &\mapsto \mathrm{id}_{\mathbb{R}} \cdot p \times \hat{z} \end{aligned}$$

where $\hat{z}$ is the unit vertical vector field on $\mathbb{R}^3$. For each $A \in \mu(\mathbb{R}^3)$, the reduced space $\mu^{-1}(A)/\mathbb{R}$ is a point.

Example 7.13. If $T \leq G$ is a maximal torus of the compact semisimple Lie group G. As above let θ be the Maurer-Cartan form on G, and let $\pi_T : \mathfrak{g} \to \mathfrak{t}$ be the projection induced by the Adjoint representation of G on $\mathfrak{g}$. Then $\omega = \mathrm{d}(\pi_T\theta)$ is a $\mathfrak{t}$-valued presymplectic form on G and the right regular action of T on (G, ω) is Hamiltonian with moment map identically zero. The reduced space is G/T and the form ω_0 is nondegenerate, thus $(G/T, \omega_0)$ is a $\mathfrak{t}$-valued symplectic space. For each $\alpha \in \mathfrak{g}^*$, the α-component of the reduced space $(G/T, \alpha \circ \omega)$ is canonically symplectomorphic to canonical symplectic structure on the coadjoint orbit $\mathcal{O}_\alpha$.

Proposition 7.5. *Let (M, ω, G, μ) be a V-valued Hamiltonian system and suppose that the reduced form $\omega_0 = 0$. Then the regular part of $\mu^{-1}(0) \subseteq M$ is a Lagrangian submanifold of M.*

Proof. Let $x \in \mu^{-1}(0)$. Since ω_x descends to the zero form on $T_x\mu^{-1}(0)/\underline{\mathfrak{g}}_x$, it follows that ω_x is the zero form on $T_x\mu^{-1}(0)$. Thus $T_x\mu^{-1}(0) \leq T_x\mu^{-1}(0)^\omega$. An application of Lemma 6.1 to the relations

$$\underline{\mathfrak{g}}_x \leq \underline{\mathfrak{g}}_x^\omega = T_x\mu^{-1}(0)$$

yields

$$T_x\mu^{-1}(0) = \underline{\mathfrak{g}}_x^\omega \geq \underline{\mathfrak{g}}_x^{\omega\omega} = T_x\mu^{-1}(0)^\omega$$

Therefore, $T_x\mu^{-1}(0) = T_x\mu^{-1}(0)^\omega$. □

We complete this chapter with a result on the topology of the reduced space.

Theorem 7.3. *Let (M, ω, G, μ) be a G-Hamiltonian system with compact Lie group G, and suppose that $0 \in \mathfrak{g}^*$ is a regular value of the moment map $\mu : M \to \mathfrak{g}^*$. Then M_0 has at most orbifold singularities.*

The proof is precisely analogous to that of the corresponding classical theorem. See Theorem 3.4 of Section 3.6.

Though we do not do so here, it is interesting to consider the topological question for more general V-Hamiltonian systems.

Chapter 8

Classical Mechanics in the V-Symplectic Setting

As briefly outlined in Section 3.1, symplectic geometry first arose in order to express and facilitate early developments in classical mechanics. It is thus a natural that we should seek to determine the extent to which the vector-valued formalism parallels the classical theory. It transpires that the analogy is very strong. Not only is there a vector-valued Hamiltonian formalism on the phase space $\mathrm{Hom}(TQ, V)$, but there is also a corresponding theory of vector-valued Lagrangian dynamics on TQ. Indeed, a suitable Lagrangian function $L : TQ \to V$ induces a V-valued symplectic structure on TQ, and yields a fiber-preserving immersion $\mathbb{F}L : TQ \to \mathrm{Hom}(TQ, V)$ as well.

In addition to the similarities, there are notable qualitative differences between the classical theory and its vector-valued counterpart. For example, in the traditional paradigm, the fiber derivative $\mathbb{F}L : TQ \to T^*Q$ of a strongly convex Lagrangian is a diffeomorphism whenever it is proper. In contrast, the fiber derivative $\mathbb{F}L : TQ \to \mathrm{Hom}(TQ, V)$ of a V-valued Lagrangian is never a diffeomorphism when $\dim V \geq 2$. Thus, we obtain a distinguished class of immersed V-symplectic submanifolds of $\mathrm{Hom}(TQ, V)$. It is interesting to observe that, as exhibited in Example 8.1 below, it is possible that the image of the fiber derivative be compact.

The phase space $\mathrm{Hom}(TQ, V)$ also provides a setting for a stronger version of the V-symplectic reduction theorem. More specifically, there is a standard moment map μ for any basic action of $\mathrm{Hom}(TQ, V)$, with respect to which the reduced space is symplectic. We leverage this property to obtain Theorem 8.4 below. Examples of V-symplectic manifolds to which the result applies are the spaces $(G, -\mathrm{d}\theta)$ of Example 7.3 and the spaces obtained via the Lagrangian formalism in Sections 8.3 and 8.4.

The interested reader may wish to consult Chapters 2 and 3 of [60] for a thorough treatment of the real-valued theory.

8.1 The V-Symplectic Structure on $\mathrm{Hom}(TQ, V)$

Our present order of business is to show that the phase space $\mathrm{Hom}(TQ, V)$ possesses a canonical V-symplectic structure which generalizes that on the cotangent bundle T^*Q. We begin with an examination of the model V-symplectic structure on the typical fiber.

Proposition 8.1. *Let U and V be vector spaces of dimension at least 1. The assignment*

$$\omega : \big(U \oplus \mathrm{Hom}(U, V)\big) \times \big(U \oplus \mathrm{Hom}(U, V)\big) \to V$$

given by

$$\omega(u + \phi, u' + \phi') = \phi'(u) - \phi(u')$$

is a V-valued symplectic structure on $U \oplus \mathrm{Hom}(U, V)$.

Proof. Let $u_1, u_2, u' \in U$, $\phi_1, \phi_2, \phi' \in \mathrm{Hom}(U, V)$, and observe that

$$\begin{aligned}
\omega\big((u_1 + \phi_1) + (u_2 + \phi_2), u' + \phi'\big) &= \phi'(u_1 + u_2) - (\phi_1 + \phi_2)(u') \\
&= \big[\phi'(u_1) - \phi_1(u')\big] + \big[\phi'(u_2) - \phi_2(u')\big] \\
&= \omega(u_1 + \phi_1, u' + \phi') + \omega(u_2 + \phi_2, u' + \phi')
\end{aligned}$$

As a similar computation shows ω to be linear in the second argument, we deduce that ω is bilinear. Since

$$\phi'(u) - \phi(u') = -\big(\phi(u') - \phi'(u)\big)$$

it follows that ω is alternating. If $u + \phi \in U \oplus \mathrm{Hom}(U, V)$ is nonzero, we can choose $u' \in U$ and $\phi' \in \mathrm{Hom}(U, V)$ so that precisely one of $\phi(u')$ and $\phi'(u)$ is nonzero. It follows that

$$\omega(u+\phi, u'+\phi') = \phi'(u) - \phi(u') \neq 0$$

from which we conclude that ω is nondegenerate. $\square$

To simplify notation, we will consider U and $\mathrm{Hom}(U,V)$ as subspaces of $U \oplus \mathrm{Hom}(U,V)$ and take $\oplus$ to denote the internal direct sum.

It is evident that U and $\mathrm{Hom}(U,V)$ are Lagrangian subspaces of $U \oplus \mathrm{Hom}(U,V)$.

Lemma 8.1. *If $W \leq \mathrm{Hom}(U,V)$, then $U \oplus W$ is symplectic precisely when the annihilator $W^0 \leq U$ is trivial.*

Proof. For $u \in U$, we have

$$u \in W^\omega \iff \forall \phi \in W : \omega(u,\phi) = \phi(u) = 0 \iff u \in \hat{U}^0$$

so that $U \cap W^\omega = U \cap W^0 = W^0$. As U is Lagrangian in $U \oplus \mathrm{Hom}(U,V)$, Lemma 6.1 yields

$$(U \oplus W)^\omega = U \cap W^\omega = W^0$$

from which we deduce

$$(U \oplus W)^\omega \cap (U \oplus W) = \hat{U}^0 \cap (U \oplus W) = W^0$$

Thus, $U \oplus W$ is symplectic if and only if $W^0 = 0$. $\square$

Lemma 8.2. *Let $T \leq U$ and $W \leq \mathrm{Hom}(U, V)$, define the subspace*

$$I(T) = \{\phi \,|\, \phi(T) = 0\} \leq \mathrm{Hom}(U, V)$$

and let $W^0 \leq U$ be the annihilator of W. Then,

(i) $T^\omega = U \oplus I(T)$

(ii) $W^\omega = W^0 \oplus \mathrm{Hom}(U, V)$

(iii) $T^{\omega\omega} = T$

Proof. (i) For $\phi \in \mathrm{Hom}(U, V)$, we have

$$\phi \in T^\omega \iff \forall u \in T : \omega(u, \phi) = \phi(u) = 0 \iff \phi \in I(T)$$

From $\omega(U, T) = 0$, it follows that

$$U \leq T^\omega \leq U \oplus \mathrm{Hom}(U, V)$$

Therefore,

$$T^\omega = U \oplus \big(T^\omega \cap \mathrm{Hom}(U, V)\big) = U \oplus I(T)$$

(ii) For any $u \in U$,

$$u \in W^\omega \iff \forall \phi \in W : \omega(u, \phi) = \phi(u) = 0 \iff u \in W^0$$

Now, $\omega\big(\mathrm{Hom}(U,V), W\big) = 0$ yields

$$\mathrm{Hom}(U,V) \leq W^\omega \leq U \oplus \mathrm{Hom}(U,V)$$

so that

$$W^\omega = (W^\omega \cap U) \oplus \mathrm{Hom}(U,V) = W^0 \oplus \mathrm{Hom}(U,V)$$

(iii) Applying parts (i) and (ii) in succession, and recalling Lemma 6.1, we obtain

$$\begin{aligned} T^{\omega\omega} &= \big(U \oplus I(T)\big)^\omega \\ &= U \cap I(T)^\omega \\ &= U \cap \big(I(T)^0 \oplus \mathrm{Hom}(U,V)\big) \\ &= I(T)^0 \end{aligned}$$

For any $u \in U$, we have

$$u \in T \implies \forall \phi \in I(T) : \phi(u) = 0 \iff u \in I(T)^0$$

Conversely, if $u \notin T$ then there is a $\phi \in I(T)$ with $\phi(u) \neq 0$, and thus $u \notin I(T)^0$.

□

Remark 8.1. There is not an analogous result for the double orthogonal $W^{\omega\omega}$ when $\dim V > 1$. While it is readily shown that $W^{\omega\omega} = I(W^0) \leq \mathrm{Hom}(U,V)$, this space can be strictly larger than W.

Let Q be a smooth manifold and suppose V is a vector space. The vector bundle $\mathrm{Hom}(TQ, V)$, with fibers $\mathrm{Hom}(T_xQ, V)$, inherits a natural V-symplectic form.

Given a smooth manifold Q, we would like to equip the tangent fibers

$$T_\phi \mathrm{Hom}(TQ, V) \cong T_qQ \oplus \mathrm{Hom}(T_qQ, V), \qquad q = \pi(\phi)$$

with a symplectic structure ω_ϕ that corresponds to ω on the right-hand side. As the identification of fibers is noncanonical, our present task is to show that ω_ϕ does not depend on our choice of splitting. To this end, we will utilize the following construction.

Definition 8.1. Define the *canonical* 1*-form* $\theta \in \Omega^1\big(\mathrm{Hom}(TQ, V), V\big)$ by

$$\theta_\phi(X) = \phi(\pi_* X)$$

where $\phi \in \mathrm{Hom}(TQ, V)$, $X \in T_\phi \mathrm{Hom}(TQ, V)$, and $\pi : \mathrm{Hom}(TQ, V) \to Q$ is the projection map.

Note that θ is well-defined as $\pi_* X \in T_xQ$ and $\phi \in \mathrm{Hom}(T_qQ, V)$, for $q = \pi(\phi)$.

Theorem 8.1. *The form* $\omega = -\mathrm{d}\theta$ *is a* V*-valued symplectic structure on* $\mathrm{Hom}(TQ, V)$*. Moreover, the form* ω_ψ *on the vector space* $T_\psi \mathrm{Hom}(TQ, V)$ *is equivalent to the canonical* V*-valued symplectic structure on* $T_\psi Q \oplus \mathrm{Hom}(T_\psi Q, V)$ *under* *any local trivialization of* $\mathrm{Hom}(TQ,V)$.

Proof. All that remains to be shown is nondegeneracy. As this is a local condition, it suffices to consider the case that Q is the model space $U=\mathbb{R}^n$. Consider the natural identification

$$T_\psi \mathrm{Hom}(TU,V) \cong U \oplus \mathrm{Hom}(U,V), \qquad \psi \in \mathrm{Hom}(TU,V)$$

and consider $u+\phi, u'+\phi' \in U \oplus \mathrm{Hom}(U,V)$ as constant tangent vector fields on $\mathrm{Hom}(TU,V)$. Under this identification, the canonical 1-form $\bar{\theta} \in \Omega^1(\mathrm{Hom}(TU,V),V)$ is given by

$$\bar{\theta}_\psi(u+\phi) = \psi(u), \qquad \psi \in \mathrm{Hom}(U,V)$$

Using the fact that constant tangent vector fields commute, we obtain

$$\begin{aligned}
\mathrm{d}\bar{\theta}_\psi(u+\phi, u'+\phi') &= \frac{\mathrm{d}}{\mathrm{d}t}\,(\psi + tu + t\phi)(u')\big|_{t=0} - \frac{\mathrm{d}}{\mathrm{d}t}(\psi + tu' + t\phi')(u)\big|_{t=0} \\
&= \phi(u') - \phi'(u) \\
&= -\omega(u+\phi, u'+\phi')
\end{aligned}$$

This establishes both claims of the theorem. $\square$

Remark 8.2. We could instead have noted that the form ω on the vector space U is invariant with respect to the natural action of $\operatorname{Aut} U$,

$$\omega\big(g\cdot(u+\phi), g\cdot(u'+\phi')\big) = \phi' g^{-1}(gu) - \phi g^{-1}(gu') = \omega(u+\phi, u'+\phi'), \qquad g \in \operatorname{Aut} U$$

and consequently that the constant form ω on the trivial bundle $\pi^*_{\mathrm{Hom}(TQ,V)}P(TQ) \times \big(U \oplus \mathrm{Hom}(U,V)\big)$ descends to a well-defined form ω on

$$T\,\mathrm{Hom}(TQ,V) \cong \pi^* P(TQ) \times_{\operatorname{Aut} U} \big(U \oplus \mathrm{Hom}(U,V)\big)$$

where we note that the identification is natural. It immediately follows that $\omega \in \Omega^2\big(\mathrm{Hom}(TQ,V),V\big)$ is fiberwise equivalent to ω and consequently is nondegenerate. That ω is closed follows readily from the fact that it is the quotient of a constant form. On the other hand, it is not clear from this argument that ω is the exterior derivative of the canonical 1-form $\theta \in \Omega^1\big(\mathrm{Hom}(TQ,V),V\big)$, a fact that we will recurrently invoke.

Remark 8.3. If $Y \in T_\phi\mathrm{Hom}(TQ,V)$ is vertical, then

$$\omega_\phi(X,Y) = Y(X)$$

where on the left-hand side we identify Y with an element of $\mathrm{Hom}(T_xQ,V)$.

Proposition 8.2. *Given $f \in \mathrm{Diff}(Q)$, with induced transformation $\bar{f} = f^*$ on*

$\mathrm{Hom}(TQ, V)$, *we have*

$$\bar{f}^*\theta = \theta$$

and

$$\bar{f}^*\omega = \omega$$

That is, θ and ω are fixed by the right action of $\mathrm{Diff}(Q)$ on $\mathrm{Hom}(TQ, V)$.

Proof. Fix $\phi \in TQ$ and $X \in T_{\bar{f}\phi}TQ$. From

$$f \circ \pi = \pi \circ \bar{f}^{-1}$$

we obtain

$$\begin{aligned}
\theta_{\bar{f}\phi}(X) &= \bar{f}\phi(\pi_* X) \\
&= \phi(f_*\pi_* X) \\
&= \phi(\pi_* \bar{f}_*^{-1} X) \\
&= \theta_\phi(\bar{f}_*^{-1} X)
\end{aligned}$$

and thus

$$\bar{f}^*\theta_\phi = \theta_{\bar{f}\phi}$$

The result follows as $\omega = -\mathrm{d}\theta$. □

Definition 8.2. A *basic function* on $\mathrm{Hom}(TQ, V)$ is the lift $\pi^* f \in C^\infty(Q, V)$ of any function $f \in C^\infty(Q, V)$.

As the next result establishes, basic functions are Hamiltonian and the associated flows restrict to linear translations on the fibers of $\mathrm{Hom}(TQ, V)$.

Proposition 8.3. *If $f \in C^\infty(Q)$, then the pullback $\pi^* f \in C^\infty\big(\mathrm{Hom}(TQ, V), V\big)$ is Hamiltonian with Hamiltonian vector field $\mathrm{d}f \in \Omega^1(Q, V)$. Here we identify $\mathrm{d}f$ with the corresponding vertical vector field on $\mathrm{Hom}(TQ, V)$.*

Proof. For any point $\phi \in \mathrm{Hom}(TU, V)$, we have

$$\mathrm{d}(\pi^* f)(X) = \mathrm{d}f(\pi_* X) = \omega_\phi(X, \mathrm{d}f_q)$$

where $x = \pi\phi$ and where we identify $\mathrm{d}f_q$ with the corresponding vertical vector at ϕ. $\square$

Now suppose that G is a Lie group with a right action $\rho : G^{\mathrm{op}} \to \mathrm{Diff}(Q)$ on Q, and consider the induced left action

$$\lambda : G \to \mathrm{Diff}\big(\mathrm{Hom}(TQ, V)\big)$$

on $\mathrm{Hom}(TQ, V)$.

Theorem 8.2. *The induced action of G on $\mathrm{Hom}(TQ, V)$ is Hamiltonian, with moment map*

$$\mu : \mathrm{Hom}(TQ, V) \to \mathrm{Hom}(\mathfrak{g}, V)$$

given by

$$\mu(\phi)(Y) = -\theta(\underline{Y}_\phi)$$

for $\phi \in \mathrm{Hom}(TQ, V)$ *and* $Y \in \mathfrak{g}$. *Moreover, the reduced space is a V-valued symplectic manifold whenever it is smooth.*

Proof. As Proposition 8.2 asserts that θ is fixed by G, it follows that

$$\mathrm{d}\,\theta(\underline{Y}) = \mathcal{L}_{\underline{Y}}\theta - \iota_{\underline{Y}}\mathrm{d}\theta = \iota_{\underline{Y}}\omega$$

for all $Y \in \mathfrak{g}$. Evaluating both sides at $-X \in T\,TQ$, we obtain

$$\mathrm{d}\mu(X)(Y) = -\big[\mathrm{d}\,\theta(\underline{Y})\big](X) = \omega(X, \underline{Y})$$

from which it follows that μ is a moment map for the action of G.

As the reduced form ω_0 on $\mu^{-1}(0)/G$ is always closed, we have only to show that it is nondegenerate. Since the image $\underline{\mathfrak{g}}_\phi \leq T_\phi\mathrm{Hom}(TQ, V)$ corresponds to the subspace

$$-(\rho_*\mathfrak{g})_q \oplus 0 \leq T_qQ \oplus \mathrm{Hom}(T_qQ, U), \qquad q = \pi\phi$$

under a coordinate-induced local trivialization of $\mathrm{Hom}(TQ, V)$, Lemma 8.2 implies that $\underline{\mathfrak{g}}_\phi^{\omega\omega} = \underline{\mathfrak{g}}_\phi$ and the result follows by Lemma 6.1. $\square$

We note that there is an equivalent characterization of this moment map, which we will call the *standard moment map* for the basic action of G, given by

$$\mu(\phi)(Y) = \phi(\rho_* Y)$$

8.2 Darboux Manifolds

We first recall a well-known theorem of Darboux.

Theorem 8.3 (Darboux)**.** *Let (M, ω) be a classical symplectic manifold. Then M is locally symplectomorphic to a cotangent bundle T^*Q for some manifold Q.*

This foundational result, which effectively resolves any question relating to the local structure of classical symplectic manifolds, motivates the following definition.

Definition 8.3. A V-symplectic manifold (M, ω) is said to be *Darboux* if M is locally symplectomorphic to a smooth subbundle of $\pi : \mathrm{Hom}(TQ, V) \to Q$ for some space Q

Evidently, any symplectic submanifold of a Darboux manifold is itself Darboux. We will show that this property is also preserved under symplectic reduction.

Definition 8.4. Let (M, ω) be a Darboux manifold, locally isomorphic to $\mathrm{Hom}(TQ, V)$. The action $\lambda : G \to \mathrm{Diff}(M)$ is said to be *basic* if λ is locally equivalent to an action $G \curvearrowright \mathrm{Hom}(TQ, V)$ which is induced by a right action $G \curvearrowleft Q$.

Definition 8.5. We will say that the V-Hamiltonian system (M', ω', G, μ') is *locally equivalent* to the V-Hamiltonian system (M, ω, G, μ) if for each point $x \in M'$ there is a symplectomorphism $f : O' \to O \subseteq M$, from a neighborhood O' of x to $O \subseteq M$, such that

(i) $\mu'|_O = f^*\mu$, and

(ii) f intertwines the action of $\mathfrak{g}$ on $\mathfrak{X}(O')$ and $\mathfrak{X}(O)$.

Observe that the relation of local equivalence of V-Hamiltonian systems is not an equivalence relation; in particular, it is not reflexive. Also note that, while we have no use for such generality, there is a natural extension of this construction for the actions of distinct groups G' and G on M' and M, respectively.

Definition 8.6. A *Hamiltonian subsystem* of (M, ω, G, μ) is any Hamiltonian system of the form $(N, \omega|_N, G, \mu|_N)$, where $N \subseteq M$.

Equipped with this terminology, we are now ready to state a version of the V-symplectic reduction theorem which ensures the nondegeneracy of the reduced 2-form.

Theorem 8.4 (Basic reduction of Darboux manifolds)**.** *If the V-Hamiltonian system (M', ω', G', μ') is locally equivalent to a Hamiltonian subsystem of $\big(\mathrm{Hom}(TQ, V), \omega, G, \mu\big)$ for a basic action of G and standard moment map μ, then the reduced space (M_0, ω_0) is Darboux. In particular, (M_0, ω_0) is symplectic.*

Proof. As local equivalence is clearly preserved under reduction, it suffices to consider a symplectic submanifold $i : M \hookrightarrow \mathrm{Hom}(TQ, V)$ and an basic action of G. As μ is the standard moment map for the action on G on $\mathrm{Hom}(TQ, V)$, it follows that $\mu^{-1}(0)$

is a vector subbundle of $\mathrm{Hom}(TQ, V)$ over Q with fibers given by

$$\begin{aligned}\mu^{-1}(0)_q &= \underline{\mathfrak{g}}_q^0 \\ &\cong \mathrm{Hom}(T_qQ/\underline{\mathfrak{g}}_q, V) \\ &\cong \mathrm{Hom}\big(T_{q\cdot G}(Q/G), V\big)\end{aligned}$$

Dividing by the action of G, we obtain a smooth equivalence

$$\mu^{-1}(0)/G \cong \mathrm{Hom}\big(T(Q/G), V\big)$$

which is readily seen to be a symplectomorphism. Now, $\mu_M = \mu \circ i$, so that

$$\mu_M^{-1}(0)/G = \mu^{-1}(0)/G \cap M/G$$

is diffeomorphic to a smooth subbundle of $\mathrm{Hom}\big(T(Q/G), V\big)$. That $\mu_M^{-1}(0)/G$ is symplectic follows by an argument similar to the proof of Theorem 8.2. Using the fact that the symplectic structure on M is the restriction of that on $\mathrm{Hom}(TQ, V)$, we conclude that the inclusion of $\mu_M^{-1}(0)/G$ in $\mathrm{Hom}\big(T(Q/G), V\big)$ is symplectic. $\square$

We present two primary examples of the basic reduction of Darboux manifolds. The first is our familiar $\mathfrak{g}$-symplectic space $(G, -\mathrm{d}\theta)$, which we establish shortly. The second example is that of manifold with a Lagrangian-induced V-symplectic form, under any Hamiltonian action of a Lie group G. These latter spaces are the subject of Sections 8.3 and 8.4.

Theorem 8.5. *Let G be a compact semisimple Lie group and $\theta \in \Omega^1(G, \mathfrak{g})$ the Maurer-Cartan form on G. Then $(G, -\mathrm{d}\theta)$ is a Darboux $\mathfrak{g}$-symplectic manifold.*

Proof. To avoid a conflict of notation, we will denote by $\alpha \in \Omega^1(G, \mathfrak{g})$ the Maurer-Cartan form, and by $\theta : \Omega^1\big(\mathrm{Hom}(TG, \mathfrak{g}), \mathfrak{g}\big)$ the fundamental 1-form on $\mathrm{Hom}(TG, \mathfrak{g})$. Consider the map

$$\begin{aligned} f : G &\to \mathrm{Hom}(TG, \mathfrak{g}) \\ g &\mapsto \alpha_g \end{aligned}$$

It follows that f is a section of $\pi : \mathrm{Hom}(TQ, \mathfrak{g}) \to G$, so that $\pi \circ f = 1$ and thus

$$\theta_{fg}(f_* X) = (fg)\big[(\alpha \circ \pi)_* X\big] = \alpha_g(X)$$

for any $g \in G$ and $X \in T_g G$. We conclude that $\alpha = f^*\theta$. $\square$

Remark 8.4. Since $\dim \mathrm{Hom}(TG, \mathfrak{g}) = \dim G(\dim G + 1)$, the map $f : G \to \mathrm{Hom}(TG, \mathfrak{g})$ is a diffeomorphism only when $\dim G = 0$.

8.3 The Lagrangian Function and the Fiber Derivative

Let U and V be vector spaces and let $L : U \to V$ be a smooth function. Under the natural identification $TU \cong U \times U$, the derivative $\mathrm{d}L : TU \to V$ yields a smooth map

$$\mathbb{F}L : U \to \mathrm{Hom}(U, V)$$

$$u \mapsto \mathrm{d}L_u$$

More concretely,

$$\mathbb{F}L(u)\, w = \mathrm{d}L_u(w) = \frac{\mathrm{d}}{\mathrm{d}t} L(u + tw)\big|_{t=0} \in V, \qquad u, w \in U$$

Note that when the dimension of V is strictly greater than 1, the fiber derivative $\mathbb{F}L$ cannot be a diffeomorphism. This constitutes a substantive difference between the classical context, in which $V = \mathbb{R}$, and the present construction for arbitrary V.

The second derivative of L at u,

$$\mathrm{d}^2 L_u : U \times U \to \mathrm{Hom}(U, V)$$

is given by

$$\mathrm{d}^2 L_u(w, w') = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} L(u + tw + sw')\big|_{s=0,t=0}, \qquad w, w' \in U$$

and defines a symmetric bilinear $\mathrm{Hom}(U, V)$-valued form on U.

Lemma 8.3. *For a smooth function $L : U \to \mathrm{Hom}(U, V)$, we have,*

(i) $\ker \mathrm{d}(\mathbb{F}L)_u = \ker \mathrm{d}^2 L_u$

(ii) $\mathbb{F}L$ *is an immersion if and only if* $\mathrm{d}^2 L_u$ *is nondegenerate at every point* $u \in U$.

(iii) The annihilator of $\mathrm{d}(\mathbb{F}L)_u(U) \leq \mathrm{Hom}(U, V)$ *is equal to the kernel of* $\mathrm{d}(\mathbb{F}L)_u : U \to \mathrm{Hom}(U, V)$.

The fiber derivative $\mathbb{F}L$ *is an immersion if and only if the second derivative is non-degenerate at every point* $u \in U$.

Proof. (i) Given $u, w \in U$, a direct computation yields

$$\begin{aligned}
\mathrm{d}(\mathbb{F}L)_u(w) = 0 &\iff \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{F}L(u + tw)\big|_{t=0} = 0 \\
&\iff \forall w' \in U \;:\; \frac{\mathrm{d}}{\mathrm{d}t}\mathbb{F}L(u + tw)(w')\big|_{t=0} = 0 \\
&\iff \forall w' \in U \;:\; \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} L(u + tw + sw')\big|_{s=0,t=0} = 0 \\
&\iff \forall w' \in U \;:\; \mathrm{d}^2 L_u(w, w') = 0
\end{aligned}$$

(ii) This follows immediately from (i).

(iii) Let $u, w \in U$. By extending the computation from (i), we obtain

$$\begin{aligned}
\mathrm{d}(\mathbb{F}L)_u(w) = 0 &\iff \forall w' \in U \,:\, \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s} L(u + tw + sw')\Big|_{s=0,t=0} = 0 \\
&\iff \forall w' \in U \,:\, \frac{\mathrm{d}}{\mathrm{d}s} \mathbb{F}L_{u+sw'}(w)\Big|_{s=0} \\
&\iff \forall w' \in U \,:\, \mathrm{d}(\mathbb{F}L)_u(w')\, w = 0
\end{aligned}$$

from which we conclude,

$$\ker \mathrm{d}(\mathbb{F}L)_u = \mathrm{d}(\mathbb{F}L)_u(U)^0$$

□

Now suppose that Q is a manifold and $L : TQ \to V$ is smooth. By applying $\mathbb{F}L$ at each fiber T_qQ, with respect to the fiber restrictions $L_q : T_qQ \to V$, we obtain a global map

$$\mathbb{F}L : TQ \to \mathrm{Hom}(TQ, V)$$

which we will call the *fiber derivative* associated to $L : TQ \to V$. As above, if $\dim V \geq 2$, the map $\mathbb{F}L$ cannot be a diffeomorphism.

Definition 8.7. Let Q be a manifold and $L : Q \to V$ a smooth map. Define the *Lagrangian 2-form* $\omega_L \in \Omega^2(TQ, V)$ to be the pullback by $\mathbb{F}L$ of the canonical V-symplectic structure $\omega \in \Omega^2\big(\mathrm{Hom}(TQ, V), V\big)$ to TQ.

Lemma 8.4. *The Lagrangian 2-form $\omega_L \in \Omega^2(TQ, V)$ is nondegenerate at $x \in TQ$ if and only if the differential*

$$\mathrm{d}(\mathbb{F}L_q)_x : T_x T_q Q \to V, \qquad q = \pi x$$

is injective.

Proof. First note that the nondegeneracy of $\mathbb{F}L^*\omega$ at x is equivalent to the condition that $\mathbb{F}L_*(T_x TQ)$ be a symplectic subspace of $T_{\mathbb{F}L(x)}\mathrm{Hom}(TQ, V)$. Since $\mathbb{F}L$ preserves fibers, we have

$$\mathbb{F}L_*(T_x TQ) \cong T_q Q \oplus \mathbb{F}L_*(T_q T_q Q)$$

with respect to any fiber trivialization $T_{\mathbb{F}L(x)}\mathrm{Hom}(TQ, V) \cong T_q Q \oplus \mathrm{Hom}(T_q Q, V)$. Now, Lemma 8.1 asserts that $T_q Q \oplus \mathbb{F}L_*(T_x T_q Q)$ is symplectic precisely when the annihilator $\mathbb{F}L_*(T_x T_q Q)^0 \leq T_q Q$ vanishes. Since $T_x T_q Q$ is tangent to $T_q Q$, an application of Lemma 8.3 yields

$$\mathbb{F}L_*(T_x T_q Q) = \mathbb{F}(L_q)_*(T_x T_q Q) = \ker \mathrm{d}(\mathbb{F}L_q)_x$$

and we conclude that $T_q Q \oplus \mathbb{F}L_*(T_x T_q Q)$ is symplectic precisely when $\ker \mathrm{d}(\mathbb{F}L_q)_x = 0$. $\square$

We are now ready to present the main result of this section.

Theorem 8.6. *The following are equivalent.*

(i) The image of $\mathbb{F}L$ *is an immersed symplectic submanifold of* $\mathrm{Hom}(TQ, V)$*.*

(ii) The Lagrangian 2*-form* ω_L *is symplectic.*

(iii) The fiber derivative $\mathbb{F}L : TQ \to \mathrm{Hom}(TQ, V)$ *is an immersion.*

(iv) The second derivative of L *is nondegenerate along the fibers of* TQ*.*

Proof. Condition (i) implies (ii) since $\omega_L = \mathbb{F}L^*\omega$.

As ω_L is the pullback of a closed 2-form, it is symplectic if and only if it is nondegenerate. Moreover, since $\mathbb{F}L$ preserves fibers, L is immersive if and only if L_q is immersive for all $q \in Q$, and thus an application of Lemma 8.4 implies that (ii) and (iii) are equivalent.

The result follows as the equivalence of (ii) and (iii) is an immediate consequence of Lemma 8.3, and since (ii) and (iii) together imply (i). □

Note that by an *immersed submanifold*, we mean the image of a manifold by an immersion.

Definition 8.8. A *V-mechanical system* (Q, L) is a pair consisting of a manifold Q and a proper map $L : TQ \to V$ which satisfies one, and hence all, of the conditions of Theorem 8.6.

Example 8.1. Consider the circle $S^1 = \mathbb{R}/\mathbb{Z}$, with tangent bundle $TS^1 \cong \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}$, and define the Lagrangian

$$\begin{aligned} L : TS^1 &\longrightarrow \mathbb{C} \\ (q, x) &\longmapsto e^{\mathrm{i}x} \end{aligned}$$

Given $q \in S^1$, and $x, y \in T_qS^1 \cong \mathbb{R}$, we have

$$\mathrm{d}(L_q)_x(y) = \frac{\mathrm{d}}{\mathrm{d}t}\, e^{\mathrm{i}(x+ty)}\Big|_{t=0} = \mathrm{i}e^{\mathrm{i}x}y$$

from which we determine that

$$\begin{aligned} \mathbb{F}L : TS^1 &\longrightarrow \mathrm{Hom}(TS^1, \mathbb{C}) \\ (q, x) &\longmapsto (q, \mathrm{i}e^{\mathrm{i}x}) \end{aligned}$$

under the identification $\mathrm{Hom}(TS^1, \mathbb{C}) \cong S^1 \times \mathbb{C}$. The image of TS^1 under $\mathbb{F}L$ is

$$\mathbb{F}L(TS^1) = S^1 \times \bar{S} \subseteq \mathrm{Hom}(TS^1, \mathbb{C}) \cong S^1 \times \mathbb{C}$$

where $\bar{S} \subseteq \mathbb{C}$ is the unit circle. Since $\mathbb{F}L$ is an immersion, it follows by Theorem 8.6 that the torus $S^1 \times \bar{S}$ is a $\mathbb{C}$-valued symplectic submanifold of $\mathrm{Hom}(TS^1, \mathbb{C})$.

8.4 Vector-Valued Lagrangian Dynamics

Let Q be a smooth manifold, let $\mathcal{C}(Q)$ be the space of paths $\tau : I \to Q$ for some fixed interval $I = [a, b] \subseteq \mathbb{R}$, and let $L : TQ \to V$ be a smooth function which we will call the *Lagrangian*. The aim of this section is to investigate those paths $\tau : I \subseteq \mathbb{R} \to Q$ which constitute the critical points of the V-valued *Lagrangian action*,

$$S(\tau) = \int_a^b L(\dot{\tau})\, \mathrm{d}t, \qquad \tau \in \mathcal{C}(Q)$$

As a matter of terminology, a *critical point* of the action S is any path $\gamma \in \mathcal{C}(Q)$ for which $\frac{\mathrm{d}}{\mathrm{d}s} S(\gamma_s)\big|_{s=0} = 0$ with respect to every variation γ_s that fixes the endpoints of γ.

Our approach is to consider instead the space $\mathcal{C}'(Q) \subseteq \mathcal{C}(TQ)$ of paths $\gamma : I \to TQ$ which arise as the velocity vector field $\dot{\tau}$ for some curve $\tau \in \mathcal{C}(Q)$. That is, $\mathcal{C}'(Q)$ is the image of the space of paths $\mathcal{C}(Q)$ by the map

$$\begin{aligned} \mathcal{C}(Q) &\longrightarrow \mathcal{C}(TQ) \\ \tau &\longmapsto \dot{\tau} \end{aligned}$$

Note that this map is injective and thus forms an equivalence between $\mathcal{C}(Q)$ and

$\mathcal{C}'(Q)$. Under this identification, we consider the Lagrangian action on $\mathcal{C}'(Q)$,

$$S(\gamma) = \int_a^b L(\gamma)\,\mathrm{d}t, \qquad \gamma \in \mathcal{C}'(Q)$$

and we aim to find the stationary points of S in $\mathcal{C}'(Q)$, under variations γ_s of γ with fixed $\pi\gamma_s(a)$ and $\pi\gamma_s(b)$.

Theorem 8.7. *Given any path $\gamma \in \mathcal{C}'(Q)$ and any variation $X : I \to T_\gamma TQ$ tangent to $\mathcal{C}'(Q)$ at γ, we have*

$$\mathrm{d}S(X) = \int_a^b \mathrm{d}\dot{L}(\pi_* X)\,\mathrm{d}t \; - \; \mathbb{F}L^*\theta(X)\big|_a^b$$

where $\dot{L} = \dot{\gamma}(L)$ is the derivative of L along γ. Consequently, $X \in T_xTQ$ is the velocity of a critical point of S at $x \in TQ$ if and only if

$$\mathrm{d}\dot{L}(\pi_* X) = 0$$

whenever this expression is well-defined.

Proof. Differentiating both sides of $\tau = \pi\gamma$ yields $\gamma = \dot{\tau} = \pi_*\dot{\gamma}$. Consequently,

$$X = \frac{\mathrm{d}}{\mathrm{d}s}\gamma_s\big|_{s=0} = \frac{\mathrm{d}}{\mathrm{d}s}\pi_*\dot{\gamma}_s\big|_{s=0} = \pi_*\dot{X}$$

and thus

$$\mathrm{d}S(X) = \int_a^b \mathrm{d}L(\pi_* \dot{X})\,\mathrm{d}t = \int_a^b \mathrm{d}\dot{L}(\pi_* X)\,\mathrm{d}t \;-\; \mathrm{d}L(\pi_* X)\big|_a^b$$

If X is tangent to a variation with fixed endpoints, then the boundary term vanishes and $\mathrm{d}\dot{L}(\pi_* X) = 0$ for all variations X tangent to $\mathcal{C}'(Q)$ along a critical point γ of the action S. The boundary term at $t = a$ is

$$\mathrm{d}L_{\gamma(a)}(\pi_* X_a) = \mathbb{F}L\big(\gamma(a)\big)(\pi_* X_a) = \mathbb{F}L^* \theta(X_a)$$

and likewise for $t = b$. □

Theorem 8.8. *Let $\gamma \in \mathcal{C}'(Q)$ be a critical point of S. Define the map*

$$H : TQ \to V$$

by

$$H(x) = \mathbb{F}L(x)\,x - L(x)$$

Then $-H$ is a Hamiltonian function for the vector field $\dot{\gamma}$ along γ.

Proof. We will adapt the approach of [61] to our V-valued context. Define the function $S : I \to V$ by

$$S(t) = \int_a^t L\big(\gamma(r)\big)\,\mathrm{d}r$$

Since $\dot\gamma$ is an infinitesimal variation of γ, Theorem 8.7 implies that

$$\mathrm{d}S(t) = \theta_L\big(\dot\gamma(a)\big) - \theta_L\big(\dot\gamma(t)\big)$$

where $\theta_L = \mathbb{F}L^*\theta$. Thus,

$$\mathcal{L}_{\dot\gamma}\theta_L = -\frac{\mathrm{d}}{\mathrm{d}t}\,\theta_L\big(\gamma(t))\big) = \frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{d}S(t) = \mathrm{d}L\big(\gamma(t)\big)$$

and so

$$-\iota_{\dot\gamma}\omega_L = \iota_{\dot\gamma}\mathrm{d}\theta_L = \mathcal{L}_{\dot\gamma}\theta_L - \mathrm{d}\iota_{\dot\gamma}\omega_L = \mathrm{d}(L - \iota_{\dot\gamma}\theta_L)$$

We conclude that $L - \iota_{\dot\gamma}\theta_L = L - \mathbb{F}L(\gamma)\,\gamma$ is a Hamiltonian function for $\dot\gamma$ along γ.

$\square$

Part III

The Moduli Space of Flat Connections

In the final part of this dissertation, we turn our attention once more to the space of connections. While Chapter 9 employs the framework of Part II, and Section 10.1 requires Chapter 3, the remaining material can be read immediately after Chapter 2 and a knowledge of only the definition of the symplectic volume.

In Chapter 9 we apply the theory of Part II to characterize the space of connections $\mathcal{A}(P)$ on a G-principal bundle P over a manifold M of dimension at least 3. In particular, we show that $A(P)$ possesses a natural vector-valued symplectic structure, that the moduli space of flat connections $\mathcal{M}(P)$ is the symplectic reduction of $\mathcal{A}(P)$ with respect to the action of the gauge group, and that the reduced form takes values in $H^2(M)$. Utilizing the language of characteristic forms, we obtains similar results for a variant of this vector-valued symplectic structure. In Chapter 10 we compute the volume of the moduli space $\mathcal{M}_G(M)$ of flat G-connections on M, first in the case that G is abelian, and second in the case that G is semisimple and $\pi_1 M$ is free abelian. In Chapter 11 we show that if $\Sigma \subseteq M$ is a distinguished embedded surface in M, then there is a symplectic immersion of the moduli space $\mathcal{M}_G(M)$ into $\mathcal{M}_G(\Sigma)$, thus yielding information on the possible structure of $\mathcal{M}_G(M)$. We refer to the second half of Section 1.1 for an outline of the main results.

Chapter 9

The Reduction of the Space of Connections

Let M be a smooth manifold of dimension greater than 2, let G be a Lie group with Ad-invariant metric $\langle\,,\rangle$ on its Lie algebra $\mathfrak{g}$, and let P be a fixed G-principal bundle on M.

Before proceeding, let us briefly review the relevant notation from Chapter 9. We denote by $\mathcal{A}(P)$ the $\Omega^1(M, \mathrm{ad}P)$-affine space of connections on P. For each $A \in \mathcal{A}(P)$, we frequently utilize the identification

$$\Omega^1(M, \mathrm{ad}P) \xrightarrow{\sim} T_A\mathcal{A}(P)$$

given by

$$\alpha_A = \frac{\mathrm{d}}{\mathrm{d}t} A + t\alpha\Big|_{t=0}, \qquad A \in \mathcal{A}(P), \alpha \in \Omega^1(M, \mathrm{ad}P)$$

where $+$ denotes the action of $\Omega^1(M, \mathrm{ad}P)$ on $\mathcal{A}(P)$. In other words, we identify $\alpha \in \Omega^1(M) \cong T_0\Omega^1(M)$ with the induced vector field $\underline{\alpha} \in \mathfrak{X}(\mathcal{A}(P))$.

The exterior covariant derivative $\mathrm{d}_A : \Omega^k(M, \mathfrak{g}) \to \Omega^{k+1}(M, \mathfrak{g})$ is given by

$$\mathrm{d}_A\sigma(X_1, \ldots, X_{k+1}) = \mathrm{d}\sigma(h_A X_1, \ldots, h_A X_{k+1} SS)$$

where $h_A : TP \to A$ is the fiberwise projection induced by the splitting $A \oplus V(P)$, where $V(P)$ is the vertical tangent bundle of P. Since d_A preserves the subspace of tensorial forms in $\Omega^*(P, \mathfrak{g})$ of type $\mathrm{Ad}P$, we may also consider $\mathrm{d}_A : \Omega^k(M, \mathrm{ad}P) \to \Omega^{k+1}(M, \mathrm{ad}P)$. Many of the results of this chapter rely on the property that

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}_A\alpha \wedge \beta + (-1)^{\deg\alpha}\alpha \wedge \mathrm{d}_A\beta$$

See [2]. Finally, we write $B^k(M)$ for the space of k-coboundaries on M. That is,

$$B^k(M) = \mathrm{d}\,\Omega^{k-1}(M) \leq \Omega^k(M)$$

9.1 The Model Space $\big(\Omega^1(M), \wedge\big)$

In this section we will consider two distinct vector-valued symplectic structures on $\Omega^1(M)$.

Let Σ be a compact oriented surface. We have seen that the vector space $\Omega^1(\Sigma)$ carries a natural symplectic structure: namely,

$$\omega(\alpha, \beta) = \int_\Sigma \alpha \wedge \beta, \qquad \alpha, \beta \in \Omega^1(\Sigma)$$

The aim of this chapter is to determine a suitable generalization of this symplectic form to the case where $\dim M \geq 3$.

The most natural vector-valued symplectic structure on $\Omega^1(M)$ is the wedge product $\wedge$. The following proposition establishes that $\wedge$ is indeed a $\Omega^2(M)$-valued symplectic form on $\Omega^1(M)$.

Proposition 9.1. *Let M be a manifold with boundary of dimension at least 2. The wedge product*

$$\wedge : \Omega^1(M) \otimes \Omega^1(M) \longrightarrow \Omega^2(M)$$

is an $\Omega^2(M)$-valued symplectic structure on $\Omega^1(M)$.

Proof. As $\wedge$ is clearly a skew-symmetric $\Omega^2(M)$-valued form on $\Omega^1(M)$, we have only to show that it is nondegenerate. Thus let $\alpha \in \Omega^1(M)$ and suppose that $\alpha \wedge \beta = 0$ for all $\beta \in \Omega^1(M)$. Let $(x^i)_{i \leq n}$ ($n = \dim M$) be a system of coordinates on a neighborhood

$U \subseteq M$, and let $\alpha_i \in C^\infty(U)$ be given by

$$\alpha = \sum_i \alpha_i \, \mathrm{d}x^i$$

For each $k \leq n$,

$$0 = \alpha \wedge \mathrm{d}x^k = \sum_i \alpha_i \, \mathrm{d}x^i \wedge \mathrm{d}x^k$$

Since $n \geq 2$, for each $i \leq n$ there is a $k \leq n$ with $k \neq i$, and thus $\mathrm{d}x^i \wedge \mathrm{d}x^k \neq 0$ so that $\alpha_i = 0$. Since our choice of U was arbitrary, we conclude that $\alpha = 0$. □

It turns out that this symplectic structure is too fine for our purposes. The action of $C^\infty(M)$ on $\Omega^1(M)$ given by

$$f \cdot \alpha = \mathrm{d}f + \alpha$$

is not in general Hamiltonian with respect to symplectic structure ω obtained by lifting $\wedge$ to the fibers of $T\Omega^1(M)$. The issue is settled by reducing the space of coefficients from $\Omega^2(M)$ to $\Omega^2(M)/B^2(M)$. To show this, we first establish a technical lemma.

Lemma 9.1. *(i) Let U be a vector space with $\dim U \geq 3$ and let $w \in \Lambda^2 U$. If $u \wedge w = 0$ for all $u \in U$ then $w = 0$.*

(ii) Let M be a manifold with $\dim M \geq 3$. If $\theta \in \Omega^2(M)$ satisfies $\mathrm{d}(f\theta) = 0$ for all $f \in C^\infty(M)$, then $\theta = 0$.

Proof. (i) Fix a basis $\{e_i\}_{i \leq n}$ ($n = \dim U$) of U and choose constants $w^{ij} \in \mathbb{R}$

$(i, j \leq n)$ so that

$$w = \sum_{i,j \leq N} w^{ij}\, e_i \wedge e_j$$

For each $k \leq n$, we have

$$0 = e_k \wedge \omega = \sum_{i,j \leq n} w^{ij} e_k \wedge e_i \wedge e_j$$

Since $n \geq 3$, for every pair of distinct $i, j \leq n$ we can find a $k \leq n$ with $k \neq i, j$. Consequently, $e_k \wedge e_i \wedge e_j \neq 0$ and thus $w^{ij} = 0$.

(ii) Since $\mathrm{d}(1 \cdot \theta) = 0$, we have

$$\mathrm{d}f \wedge \theta = \mathrm{d}(f\theta) = 0$$

for all $f \in C^\infty(M)$. Fix $p \in M$ and observe that $\alpha \wedge \theta_p = 0 \in \Lambda^3(T_p^* M)$ for all $\alpha = \mathrm{d}f_p \in T_p^* M$. Now part (i) yields $\theta_p = 0$.

$\square$

Proposition 9.2. *Let M be a compact manifold of dimension at least 3. The assignment*

$$\omega : \Omega^1(M) \otimes \Omega^1(M) \longrightarrow \Omega^2(M)/B^2(M)$$

defined by

$$\omega(\alpha, \beta) = \alpha \wedge \beta + B^2(M), \qquad \alpha, \beta \in \Omega^1(M)$$

is an $\Omega^2(M)/B^2(M)$*-valued symplectic structure on* $\Omega^1(M)$ *if and only if* $\dim M \geq 3$ *or* M *is a closed orientable surface.*

Proof. The cases where $\dim M = 0, 1$ are clear. Suppose for the moment that $\dim M \geq 3$. Let $\alpha \in \Omega^1(M)$ and assume that $\alpha \wedge \gamma \in B^2(M)$ for all $\gamma \in \Omega^1(M)$. Let $\beta \in \Omega^1(M)$ and observe that

$$\mathrm{d}(\alpha \wedge f\beta) = \mathrm{d}\big[f(\alpha \wedge \beta)\big] \in B^2(M)$$

for all $f \in C^\infty(M)$. Thus $\alpha \wedge \beta = 0$ by Lemma 9.1. Since our choice of β was arbitrary, the nondegeneracy of the wedge product yields $\alpha = 0$.

Finally, suppose that $M = \Sigma$ is a closed compact orientable surface and equip Σ with an orientation and a Riemannian structure g. Let $* : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$ denote the Hodge star operator. Let $\alpha \in \Omega^1(\Sigma)$ and observe that if

$$\alpha \wedge \beta \in B^2(\Sigma)$$

for all $\beta \in B^2$, then, in particular,

$$\|\alpha\|^2 \,\mathrm{dvol} = \alpha \wedge *\alpha \in B^2(\Sigma)$$

and thus

$$\int_\Sigma \|\alpha\|^2 \,\mathrm{dvol} = 0$$

so that $\alpha = 0$, as required.

If $M = \Sigma$ is a connected surface which is nonorientable, noncompact, or has nonempty boundary, then the space of coefficients $\Omega^2(\Sigma)/B^2(\Sigma) = 0$ is trivial and ω is the zero form. The case for disconnected Σ is similar. $\square$

9.2 Cohomology as $\Omega^2(M)/B^2(M)$-Symplectic Reduction

Let M be a smooth manifold with boundary, let ω be the $\Omega^2(M)/B^2(M)$-valued 2-form on $\Omega^1(M)$ be given by

$$\omega_A(\alpha, \beta) = \alpha \wedge \beta + B^2(M), \qquad A \in \Omega^1(M),\ \alpha, \beta \in \Omega^1(M) \cong T_A\Omega^1(M)$$

and let $C^\infty(M)$ act on $\Omega^1(M)$ by

$$f \cdot \alpha = \mathrm{d}f + \alpha$$

The aim of this section is to show that the symplectic reduction of $\big(\Omega^1(M), \omega\big)$ is the first cohomology $H^1(M)$ with the wedge product $\wedge_{H^1}$.

Proposition 9.3. *The space $\big(\Omega^1(M), \omega\big)$ is an $\Omega^2(M)/B^2(M)$-valued symplectic manifold.*

Proof. The form ω is closed as it is constant on $\Omega^2(M)$. For every $A \in \Omega^1(M)$, Proposition 9.2 ensures that the restriction of ω to the fiber $\Omega^1(M) \cong T_A\Omega^1(M)$ of the tangent bundle $T\Omega^1(M)$ is nondegenerate. This completes the proof. □

Theorem 9.1. *The action of $C^\infty(M)$ on $\Omega^1(M)$, given by $f \cdot \alpha = \mathrm{d}f + \alpha$, is Hamiltonian with respect to ω. A moment map*

$$\mu : \Omega^1(M) \longrightarrow \mathrm{Hom}\big(C^\infty(M),\, \Omega^2(M)/B^2(M)\big)$$

is given by

$$\mu(A)(f) = \mathrm{d}A \wedge f + B^2(M)$$

The reduced space is $\big(H^2(M), \omega_0\big)$ where, for each $\bar{A} \in H^1(M)$, the reduced symplectic form

$$\omega_0 : T_{\bar{A}}H^2(M) \otimes T_{\bar{A}}H^2(M) \longrightarrow H^2(M) \leq \Omega^2(M)/B^2(M)$$

is given by

$$\omega_0\big(\bar{\alpha}_{\bar{A}}, \bar{\beta}_{\bar{A}}\big) = \bar{\alpha} \wedge \bar{\beta}, \qquad \bar{\alpha}, \bar{\beta} \in T_{\bar{A}}H^2(M) \cong H^2(M)$$

Proof. For $\alpha \in \Omega^1(M)$, $f \in C^\infty(M)$, the equality

$$\mathrm{d}(\alpha \wedge f) = \mathrm{d}\alpha \wedge f - \alpha \wedge \mathrm{d}f$$

implies that

$$\big[\mathrm{d}\alpha \wedge f\big]_{\Omega^2/B^2} = \big[\alpha \wedge \mathrm{d}f\big]_{\Omega^2/B^2}$$

Since $\mathrm{d} : \Omega^1(M) \to \Omega^2(M)$ is linear, the induced map $\mathrm{d}_* : T\Omega^1(M) \to T\Omega^2(M)$ is given by

$$\mathrm{d}_*\alpha_A = (\mathrm{d}\alpha)_{\mathrm{d}A} \in T_{\mathrm{d}A}\Omega^2(M)$$

for every $A \in \Omega^1(M)$. Thus,

$$\begin{aligned} \big\langle \mathrm{d}_*\alpha_A, f \big\rangle &= \mathrm{d}\alpha \wedge f + B^2(M) \\ &= \alpha \wedge \mathrm{d}f + B^2(M) \\ &= \omega(\alpha_A, \underline{f}_A) \end{aligned}$$

and we conclude that $\mathrm{d} : \Omega^1(M) \to \Omega^2(M)$ is a moment map for the action of $C^\infty(M)$ on $\Omega^1(M)$. Since

$$\mu(A) = 0 \iff \forall f \in C^\infty(M) \,:\, \mathrm{d}A \wedge f + B^2 = 0 \iff \mathrm{d}A = 0$$

we conclude that the reduced space is $\mu^{-1}(0)/C^\infty(M) = Z^2(M)/B^2(M) = H^2(M)$.

$\square$

9.3 The Reduction of the Space of Connections

Let M be a smooth manifold with boundary, G a compact connected Lie group and let $\langle\,,\rangle$ be an Ad-invariant metric on the Lie algebra $\mathfrak{g}$, P a G-principal bundle on M, $\langle\,,\rangle_{\mathfrak{g}}$ an Ad-invariant metric on the Lie algebra $\mathfrak{g}$, and $\mathcal{A}(P)$ the $\Omega^1(M,\mathrm{ad}P)$-affine space of G-connections on P. Recall that $\langle\,,\rangle_{\mathfrak{g}}$ induces a metric $\langle\,,\rangle_{\mathrm{ad}P}$ on the fibers of $\mathrm{ad}P$, and the wedge product on $\Omega^*(M,\mathrm{ad}P)$ is given as the composition

$$\wedge : \Omega^*(M,\mathrm{ad}P)\otimes\Omega^*(M,\mathrm{ad}P) \xrightarrow{\wedge_{\Omega^*(M)}} \Omega^*(M,\mathrm{ad}P\otimes\mathrm{ad}P) \xrightarrow{\langle\,,\rangle_{\mathrm{ad}P}} \Omega^*(M)$$

Define the 2-form $\omega \in \Omega^2\big(\mathcal{A}(P),\Omega^2(M)/B^2(M)\big)$ by

$$\omega_A(\alpha,\beta) = \alpha\wedge\beta + B^2(M) \in \Omega^2(M)/B^2(M)$$

for $A\in\mathcal{A}(P)$ and $\alpha,\beta\in\Omega^2(M,\mathrm{ad}P)\cong T_A\mathcal{A}(P)$.

Theorem 9.2. *The form ω is a $\Omega^2(M)/B^2(M)$-valued symplectic structure on $\mathcal{A}(P)$ if and only if $\dim M = 0$, $\dim M \geq 3$, or M is a closed compact orientable surface.*

Proof. Closedness follows from the fact that ω is constant on $\mathcal{A}(P)$. It remains to prove ω is nondegenerate. Again, the cases where $\dim M = 0, 1$ are clear.

First suppose $\dim M \geq 3$. Fix $A\in\mathcal{A}(P)$ and assume that $\alpha\in\Omega^1(M,\mathrm{ad}P)\cong T_A\mathcal{A}(P)$ is nonzero at $x\in M$. Let $U\subseteq M$ be a closed trivializing neighborhood for ad$P$ containing x, so that U is a submanifold with boundary and

$$\Omega^1(U, \mathrm{ad}P) \cong \Omega^1(U, \mathfrak{g}) \cong \Omega^1(U)^{\dim \mathfrak{g}}$$

As α is nonzero on U, Proposition 9.2 yields a $\beta \in \Omega^1(U, \mathrm{ad}P)$ with

$$\alpha|_U \wedge \beta \notin B^2(U)$$

Since U is closed, there is an extension $\bar{\beta} \in \Omega^1(M, \mathrm{ad}P)$ of β to M, and $B^2(U) = B^2(M)|_U$. Therefore,

$$\alpha \wedge \beta \notin B^2(M)$$

and we conclude that ω is nondegenerate.

Now suppose $M = \Sigma$ is a closed compact orientable surface. Equip Σ with an orientation and a Riemannian structure g. Let $* : \Omega^1(\Sigma, \mathrm{ad}P) \to \Omega^1(\Sigma, \mathrm{ad}P)$ denote the Hodge star operator determined by g and $\langle\, , \rangle_{\mathrm{ad}P}$. If $\alpha \in \Omega^2(\Sigma, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$ satisfies

$$\alpha \wedge \beta \in B^2(\Sigma)$$

for all $\beta \in \Omega^2(\Sigma, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$, then

$$\int_\Sigma \|\alpha\|^2 \,\mathrm{dvol} = \int_\Sigma \alpha \wedge *\alpha = 0$$

and thus $\alpha = 0$. It follows that ω is nondegenerate.

Now suppose that $M = \Sigma$ is a surface which is nonorientable, noncompact, or has nonempty boundary If Σ is connected, then $\Omega^2(\Sigma)/B^2(\Sigma) = 0$ and thus $\omega \in \Omega^2(\mathcal{A}(P), \Omega^2/B^2)$ is the zero form. The case for disconnected Σ is similar. $\square$

Theorem 9.3. *Let M be either a smooth manifold of dimension at least 3, or a compact orientable surface. Fix a G-principal bundle P on M and let $\mathcal{A}(P)$ be the space of connections on P. The natural pairing*

$$\mu : \mathcal{A}(P) \longrightarrow \mathrm{Hom}\big(\mathfrak{g}, \Omega^2/B^2\big)$$

given by

$$\mu(A)(f) = F(A) \wedge f + B^2(M)$$

where $F : \mathcal{A}(P) \to \Omega^2(M, \mathrm{ad}P)$ is the curvature, is a moment map for the action of the gauge group $\mathcal{G}(P)$ on $\mathcal{A}(P)$ with respect to the symplectic structure $\omega \in \Omega^2(M)$, defined by

$$\omega(\alpha, \beta) = \alpha \wedge \beta + B^2(M)$$

for $\alpha, \beta \in \Omega^1(M, \Omega^2/B^2)$. The reduced space $\mathcal{A}(P)_0$ is the moduli space of flat connection $\mathcal{M}(P) = F^{-1}(0)/\mathcal{G}$ on P, and the reduced form ω_0 takes values in the finite-dimensional vector space $H^2(M)$.

Proof. Fix $A \in \mathcal{A}(P)$, $\alpha \in \Omega^1(M, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$ and $f \in \Omega^0(M, \mathrm{ad}P) \cong \mathfrak{g}(P)$.

Since

$$\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}_A\alpha \wedge \beta - \alpha \wedge \mathrm{d}_A\beta$$

we have

$$\big[\mathrm{d}_A\alpha \wedge f\big]_{\Omega^2/B^2} = \big[\alpha \wedge \mathrm{d}_A f\big]_{\Omega^2/B^2}$$

As $F_*\alpha = \mathrm{d}_A\alpha$ and $\underline{f}_A = \mathrm{d}_A f$, we obtain

$$\langle \mu_*\alpha, f\rangle = \omega_A\big(\alpha, \underline{f}_A\big)$$

Therefore, F is a moment map. For any $A \in \mathcal{A}(P)$, we have

$$\mu(A) = 0 \iff \forall f \in \Omega^0(M, \mathrm{ad}P) \,:\, F(A) \wedge f \in B^2 \iff F(A) = 0$$

and we conclude that $\mu^{-1}(0)/\mathcal{G} = F^{-1}(0)/\mathcal{G}$.

If $A \in \mu^{-1}(0)$ and $\alpha, \beta \in \Omega^1(M, \mathrm{ad}P) \cong T_A\mathcal{A}$ are tangent to $\mu^{-1}(0)$, then $\mathrm{d}_A\alpha = F_*\alpha = 0$ and so $\alpha \wedge \beta \in Z^2(M)$ is a cocycle. It follows that the reduced form ω_0 on $\mathcal{M}(P)$ takes values in $Z^2(M)/B^2(M) = H^2(M)$. $\square$

Remark 9.1. Suitable adjustments are required when the base space M is noncompact, since in this case the Banach-manifold structure of $\mathcal{A}(P)$ encounters complications. This difficulty can be overcome by, for example, replacing $\mathcal{A}(P)$ with the subspace of asymptotically flat connections on P. We will not address such considerations in this exposition and, in the following, we will implicitly assume M to be compact.

Corollary 9.1. *If $H^2(M) = 0$, then Theorem 9.3 and Proposition 7.5 together imply that regular part of space of flat connections $F^{-1}(0) \subseteq \mathcal{A}(P)$ is a Lagrangian submanifold of $\mathcal{A}(P)$.*

Proof. If the second cohomology $H^2(M)$ vanishes, then the reduced form $\omega_0 \in \Omega^2\big(\mathcal{M}(P), H^2(M)\big)$ is necessarily zero, and Proposition 7.5 implies that $\mu^{-1}(0)$ is a Lagrangian submanifold of $\mathcal{A}(P)$. The result follows as $F^{-1}(0) = \mu^{-1}(0)$. $\square$

Remark 9.2. If $M = \Sigma$ is a surface, $A \in \mathcal{A}(P)$, $\alpha \in \Omega^1(\Sigma, \mathrm{ad}P) \cong T_A\mathcal{A}$, and $f \in \Omega^0(\Sigma, \mathrm{ad}P) \cong \mathfrak{g}$, then the equality

$$0 = \mathrm{d}(\alpha \wedge f) = \mathrm{d}_A\alpha \wedge f - \alpha \wedge \mathrm{d}_A f$$

implies that the function $\mu : M \to \mathrm{Hom}\big(\mathfrak{g}, \Omega^2(M)\big)$ given by

$$\mu(A)(f) = F(A) \wedge f = \mathrm{d}_A^2 f$$

is a moment map for the action of $\mathcal{G}$ on $\mathcal{A}(P)$ with respect to the $\Omega^2(\Sigma)$-valued symplectic form ω given by

$$\omega_A(\alpha, \beta) = \alpha \wedge \beta$$

The reduced space $\mathcal{A}(P)_0$ is again the moduli space of flat connections $\mathcal{M}(P)$. However, the reduced form ω_0 takes values in the infinite-dimensional vector space $Z^2(M)$ of 2-cocycles on M.

9.4 Characteristic Forms of Degree 2 and Ricci Curvature

Let M be closed manifold with $\dim M \geq 2$, let G be a Lie group, and let P be a G-principal bundle on M. We first recall a formula from [42].

Lemma 9.2 ([42] II.5.5)**.** *Let $A \in \mathcal{A}(P)$ be a connection, let $\eta \in \Omega^1(P, \mathfrak{g})$ be the connection 1-form for A, and let $\alpha \in \Omega^1(P, \mathfrak{g})$ be a tensorial 1-form of type $\mathrm{Ad}\, G$. Then*

$$\mathrm{d}_A\alpha(X,Y) = \mathrm{d}\alpha(X,Y) + \frac{1}{2}\big[\alpha(X), \eta(Y)\big] + \frac{1}{2}\big[\eta(X), \alpha(Y)\big]$$

for $X, Y \in T_uP$, $u \in P$.

This will be a key tool in the following result.

Lemma 9.3. *Suppose that $\alpha \in \Omega^1(M, \mathrm{ad}P)$. If $\phi \in \mathfrak{g}^*$ is invariant under the coadjoint action of G, then*

(i) the assignment

$$\begin{aligned} \mathrm{ad}P &\longrightarrow \mathbb{R} \\ [u, Y] &\longmapsto \phi(Y) \end{aligned}$$

is well-defined and fiberwise linear. We denote this assignment, as well as the induced maps $\Omega^k(M, \mathrm{ad}P) \to \Omega^k(M)$, by ϕ.

(ii) $\phi(\mathrm{d}_A\alpha) = \mathrm{d}(\phi\alpha)$

(iii) $\mathrm{d}(\phi\alpha \wedge \phi\beta) = \phi\, \mathrm{d}_A\alpha \wedge \phi\beta - \phi\alpha \wedge \phi\, \mathrm{d}_A\beta$

Proof. (i) Let (u, Y) and $(u', Y') \in P \times \mathfrak{g}$ represent the same element in $\mathrm{ad}P = P \times_{\mathrm{Ad}} G$. That is, we suppose there is a $g \in G$ with $u' = ug^{-1}$ and $Y' = \mathrm{Ad}_g Y$. Since ϕ is Ad^*-invariant, $\phi(Y) = \phi(Y')$ and $\bar{\phi}$ is well defined.

(ii) Let $Y, Z \in \mathfrak{g}$ be arbitrary and observe that

$$\phi[Y, Z] = \frac{\mathrm{d}}{\mathrm{d}t}\, \phi\, \mathrm{Ad}_{\exp(tY)} Z\big|_{t=0} = 0$$

Thus, by Lemma 9.2 and the linearity of ϕ, we obtain

$$\phi(\mathrm{d}_A\alpha) = \phi(\mathrm{d}\alpha) + \frac{1}{2}\phi[\alpha, \omega] + \frac{1}{2}\phi[\omega, \alpha] = \mathrm{d}(\phi\,\alpha)$$

as required.

(iii) By part (ii), we have

$$\begin{aligned} \mathrm{d}(\phi\alpha \wedge \phi\beta) &= \mathrm{d}\phi\alpha \wedge \phi\beta - \phi\alpha \wedge \mathrm{d}\phi\beta \\ &= \phi\, \mathrm{d}_A\alpha \wedge \phi\beta - \phi\alpha \wedge \phi\, \mathrm{d}_A\beta \end{aligned}$$

□

Remark 9.3. For any connection $A \in \mathcal{A}(P)$, the image of the curvature F_A under the induced map $\phi : \Omega^2(M, \mathrm{ad}P) \to \Omega^2(M)$ represents the characteristic class corresponding to the Ad-invariant multilinear map $\phi : \mathfrak{g} \to \mathbb{R}$. That is, $[\phi F_A]_{H^2}$ is the image of $\phi : \mathfrak{g} \to \mathbb{R}$ under the Chern-Weil homomorphism. We will call ϕF_A the *characteristic form* of A associated to ϕ.

Corollary 9.2. *Let V be a real (resp. complex) vector space, M a smooth manifold, E a V-vector bundle over M with structure group $\mathrm{GL}(V)$, and $\mathrm{tr} : \mathfrak{gl}(V) \to \mathbb{R}$ (resp. $\mathrm{tr} : \mathfrak{gl}(V) \to \mathbb{C}$) the trace map on $\mathfrak{gl}(V) \cong \mathrm{End}(V)$. We have*

$$\mathrm{tr}(\mathrm{d}_A\alpha) = \mathrm{d}(\mathrm{tr}\,\alpha)$$

and

$$\mathrm{d}(\mathrm{tr}\,\alpha \wedge \mathrm{tr}\,\beta) = \mathrm{tr}\,\mathrm{d}_A\alpha \wedge \mathrm{tr}\,\beta - \mathrm{tr}\,\alpha \wedge \mathrm{tr}\,\mathrm{d}_A\beta$$

We are now ready to present the main result of this section.

Theorem 9.4. *Let M be a manifold with $\dim M \geq 3$, G a Lie group with $\dim G \geq 2$, P a G-principal bundle on M, and suppose that $\phi \in \mathfrak{g}^*$ is nonzero and Ad^*-invariant. The assignment*

$$(\omega_\phi)_A(\alpha, \beta) = \phi\alpha \wedge \phi\beta + B^2(M), \qquad A \in \mathcal{A}(P),\ \alpha, \beta \in \Omega^1(M, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$$

defines a closed 2-form

$$\omega_\phi \in \Omega^2\big(\mathcal{A}(P),\, \Omega^2(M)/B^2(M)\big)$$

on the space $\mathcal{A}(P)$ of connections on P. The kernel of ω_ϕ at $A \in T_A\mathcal{A}(P)$ is

$$\ker\,(\omega_\phi)_A = \ker\big[\phi : \Omega^1(M, \mathrm{ad}P) \to \Omega^1(M)\big]$$

Moreover, the action of the gauge group $\mathcal{G}$ on $(\mathcal{A}, \omega_\phi)$ is Hamiltonian, with moment map

$$\mu_\phi : \mathcal{A}(P) \to \Omega^2(M)/B^2(M)$$

given by

$$\mu_\phi(A)Y = \phi F(A) \wedge \phi Y + B^2(M), \qquad X \in \Omega^0(M, \mathrm{ad}P)$$

and the reduced space is

$$\mathcal{A}(P)_0 = (\phi F)^{-1}(0)/\mathcal{G}$$

Proof. Closedness follows as ω_ϕ is constant on $\mathcal{A}(P)$.

Fix a connection $A \in \mathcal{A}(P)$ and a tangent vector $\alpha \in \Omega^1(M, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$. If $\alpha \in \ker\phi$, then it immediately follows that $\alpha \in \ker\omega_\phi$. If, on the other hand, $\alpha \in \ker\omega_\phi$ then

$$\phi\alpha \wedge \phi\beta \in B^2(M)$$

for all $\beta \in \Omega^1(M, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$. Since $\phi : \Omega^1(M, \mathrm{ad}P) \to \Omega^1(M)$ is surjective and $\wedge_{\Omega^1(M)}$ is nondegenerate, we deduce that $\phi\alpha = 0$. Thus, $\ker(\omega_\phi)_A = \ker\phi$.

For $f \in \Omega^0(M, \mathrm{ad}P) \cong \mathfrak{g}$ and $\alpha \in \Omega^1(M, \mathrm{ad}P)$, Lemma 9.3 implies that

$$\begin{aligned}
\big\langle (\mu_\phi)_*\alpha_A, f \big\rangle &= \phi \mathrm{d}_A\alpha \wedge \phi f + B^2(M) \\
&= \phi\alpha \wedge \phi \mathrm{d}_A f + B^2(M) \\
&= \omega(\alpha_A, \underline{f}_A)
\end{aligned}$$

and thus μ_A is a moment map for the action of $\mathcal{G}$ on $\mathcal{A}(P)$. Finally,

$$\mu_\phi(A) = 0 \iff \forall f \in \Omega^0(M, \mathrm{ad}P) \,:\, \phi F(A) \wedge \phi f \in B^2 \iff F(A) \in \ker\phi$$

so that $\mu_\phi^{-1}(0)/\mathcal{G} = (\phi F)^{-1}(0)/\mathcal{G}$. $\square$

Consider a complex manifold M and a holomorphic vector bundle E over M. Recall that the first Chern class $c_1(E)$ is represented by the form

$$c_1(A) = \frac{-1}{2\pi \mathrm{i}} \operatorname{tr} F_A \in \Omega^2(M)$$

where tr denotes the complex trace of $F_A \in \Omega^2(M, \mathrm{End}_{\mathbb{C}}E)$, and where A is any connection on the holomorphic frame bundle PE. We will call $c_1(A)$ the *first Chern form* of A. If A is the Chern connection of a hermitian structure $h : E \otimes \overline{E} \to \mathbb{C}$, then $c_1(A)$ is proportional to the Ricci form $\rho(h)$ of h [43]. This motivates the following terminology,

Definition 9.1. The connection $A \in \mathcal{A}(E)$ is said to be *Ricci flat* if $c_1(A) = 0$.

The following corollary follows immediately from

Corollary 9.3. *Let M be a complex manifold and let E be a holomorphic vector bundle over M with $c_1(E) = 0$. The moduli space of Ricci flat connections is the symplectic reduction of the space of connections $\mathcal{A}(E)$ equipped with the symplectic form ω_{tr} and moment map given by $A \mapsto \operatorname{tr} F_A$.*

Remark 9.4. Consider the map $f : \mathrm{Met}(E) \to \mathcal{A}(PE)$ from the space of hermitian structure on E to the space of connections on PE, which sends a hermitian structure h to its Chern connection $f(h)$. Then f is equivariant under the action of the gauge group, $f^*\omega_{\mathrm{tr}}$ is an $\Omega^2(M)/B^2(M)$-valued 2-form on $\mathrm{Met}(E)$, and the symplectic reduction of $\big(\mathrm{Met}(E), f^*\omega_{\mathrm{tr}}\big)$ with respect to the moment map $f^*\mu_{\mathrm{tr}}$ is the moduli space of Ricci flat hermitian structures on E.

In the case that $E = TM^{\mathbb{C}}$ is the complexified tangent bundle, then the reduced space is the moduli space of Ricci flat Kähler metrics on M.

Remark 9.5. It is significant in the preceding material that tr denotes the complex trace. Indeed, the argument cannot be adapted to Riemannian structures as $\operatorname{tr}_{\mathbb{R}} F_A = 0$ for any metric connection A.

Chapter 10

The Volume of the Moduli Space

Let (M^{2n}, η) be a symplectic manifold, G a compact Lie group with Ad-invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$, and P a G-principal bundle on M. In section 10.1, we will use the symplectic form $\eta \in \Omega^2(M)$ to define a classical symplectic structure $\omega \in \Omega^2\big(\mathcal{A}(P)\big)$ on the space of connections $\mathcal{A}(P)$, and to show that $\frac{1}{(n-1)!} F \wedge \eta^{n-1}$ is a moment map for the action of the gauge group $\mathcal{G}$ on $\mathcal{A}(P)$ with respect to ω. This construction is well-known and already appears in the literature; see, for example, [76].

The remainder of this chapter, which comprises original scholarship, is devoted to computing the volume of the moduli space $\mathcal{M}_G(M)$ of flat G-connections on M with respect to the induced symplectic structure on $\mathcal{M}_G(M)$. While the volume of the moduli space $\mathcal{M}_G(\Sigma)$ over a surface Σ has seen active progress in recent decades, see, for example, [34, 52, 53, 78, 79, 99, 100], the case for a higher dimensional base M has received little attention.

Recall that, in Theorem 8.4, we established that the holonomy map establishes a diffeomorphism on the regular parts of the moduli space $\mathcal{M}_G(M)$ and the representation variety $\mathrm{Hom}(\pi_1 M, G)/G$. For each generating set $\{\gamma_i\}_i$ of $\pi_1 G$, there is a natural corresponding realization of $\mathrm{Hom}(\pi_1 M, G)$, and hence $\mathrm{Hom}(\pi_1 M, G)/G$, as a manifold with singularities. As there is no convenient representation in terms of representatives of $\mathcal{M}_G(M)$, the question naturally arises as to the purpose of retaining the construction $\mathcal{M}_G(M)$ at all; why not work entirely with the more accessible representation variety? The relevant answer in our present context is that, unlike the moduli space $\mathcal{M}_G(M)$ for a Lefschetz symplectic manifold (M, η), the space $\mathrm{Hom}(\pi_1 M, G)/G$ does not possess a canonical volume form independent of the choice of generating set $\{\gamma_i\}_i$. We refer to [24] for an analysis of the volume in the surface case, and [70] for a discussion of volume forms on character varieties in general.

Another application that expressly invokes the symplectic structure on $\mathcal{M}_G(M)$ is the proof that the induced action of the mapping class group of Σ on $\mathcal{M}_G(\Sigma)$ is ergodic with respect to the symplectic volume [25].

While they do not address the moduli space of flat connections, the papers [19,20] may also be of interest as they investigate the extension of gauge theoretic techniques to higher dimensions.

10.1 The Classical Symplectic Structure on the Space of Connections

For notational convenience, throughout this section we assume that $\dim M = 4$ and that P is trivial.

We will regularly pair elements of $\Omega^*(M, \mathfrak{g})$ with those of $\Omega^*(M)$. This operation can be formalized by considering both spaces to be included in $\Omega^*(M, T\mathfrak{g})$, the space of differential forms with values in the tensor algebra $T\mathfrak{g}$. We will adopt the convention, however, that for any $\alpha, \beta \in \Omega^*(M, \mathfrak{g})$ the form $\alpha \wedge \beta$ will will be implicitly followed by the metric $\langle \cdot, \cdot \rangle$ on the $\mathfrak{g} \otimes \mathfrak{g}$ component, and will thus lie in $\Omega^*(M, \mathbb{R})$ and not in $\Omega^*(M, \mathfrak{g} \otimes \mathfrak{g})$.

Let $\alpha, \beta \in \Omega^1(M, \mathfrak{g})$ and define the bilinear form $\omega : \Omega^1(M, \mathfrak{g}) \otimes \Omega^1(M, \mathfrak{g}) \to \mathbb{R}$ by

$$\omega(\alpha, \beta) = \int_M \alpha \wedge \beta \wedge \eta$$

Proposition 10.1. *The form ω is a classical symplectic structure on the vector space $\Omega^1(M, \mathfrak{g})$.*

Proof. As ω is clearly bilinear and antisymmetric, it remains to show that it is non-degenerate.

Choose $\beta \in \Omega^1(M, \mathfrak{g})$ with $\beta \neq 0$. Let $U \subseteq M$ be an open set such that

(i) The form β is nonvanishing on U; and

(ii) There is a system of symplectic coordinates $(x, y, s, t) : U \to \mathbb{R}^4$, explicitly, $\eta = \mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}s \wedge \mathrm{d}t$.

Now write

$$\beta = f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}s + k\,\mathrm{d}t, \qquad f, g, h, k \in C^\infty(U, \mathfrak{g})$$

and observe that

$$\begin{aligned} \beta \wedge \eta &= (f\,\mathrm{d}x + g\,\mathrm{d}y + h\,\mathrm{d}s + k\,\mathrm{d}t) \wedge (\mathrm{d}x \wedge \mathrm{d}y + \mathrm{d}s \wedge \mathrm{d}t) \\ &= f\ \mathrm{d}x \wedge \mathrm{d}s \wedge \mathrm{d}t + g\ \mathrm{d}y \wedge \mathrm{d}s \wedge \mathrm{d}t + h\ \mathrm{d}s \wedge \mathrm{d}y \wedge \mathrm{d}x + k\ \mathrm{d}t \wedge \mathrm{d}x \wedge \mathrm{d}y \end{aligned}$$

is nonvanishing on U, for $(\beta \wedge \eta)(p) = 0$ implies $f(p) = \cdots = k(p) = 0$ and thus $\beta(p) = 0$. Suppose then, without loss of generality, that $g \neq 0$. Let $f \in C^\infty(M)$ satisfy both $f \geq 0$ and $\mathrm{supp}(f) \subseteq U$, and put $\alpha = fg\,\mathrm{d}x$. Then

$$\begin{aligned} \omega(\alpha, \beta) &= \int_U fg\,\mathrm{d}x \wedge \beta \wedge \eta \\ &= \int (fg\,\mathrm{d}x) \wedge (g\ \mathrm{d}y \wedge \mathrm{d}s \wedge \mathrm{d}t) \\ &= \int f\,\langle g, g\rangle\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}s \wedge \mathrm{d}t \\ &= \int_U f\,\langle g, g\rangle\,\mathrm{dvol}_\eta \\ &> 0 \end{aligned}$$

as required. $\square$

As P is trivial, it follows that the space $\mathcal{A}(P)$ of connections on P is an affine space modeled on $\Omega^1(M, \mathfrak{g})$. Consequently, for any $A \in \mathcal{A}(P)$, the tangent space $T_A\mathcal{A}(P)$ is canonically isomorphic to $\Omega^1(M, \mathfrak{g})$. Define the the 2-form $\omega \in \Omega^2\big(\mathcal{A}(P)\big)$ by

$$\omega_A(\alpha, \beta) = \int_M \alpha \wedge \beta \wedge \eta$$

for $A \in \mathcal{A}(P)$ and $\alpha, \beta \in \Omega^1(\Sigma, \mathrm{ad}P) \cong T_A\mathcal{A}(P)$.

Proposition 10.2. *The form ω is a symplectic structure on the space of connections $\mathcal{A}(P)$.*

Proof. It follows from Propostion 10.1 that ω is nondegenerate. We will show that it is closed.

Fix $\alpha_0, \beta_0, \gamma_0 \in \Omega^1(M)$ and let $\alpha \in \mathfrak{X}\big(\mathcal{A}(P)\big)$ be given by

$$\alpha_A(f) = \frac{\mathrm{d}}{\mathrm{d}s} f(A + s\alpha_0)\Big|_{s=0}, \qquad A \in \mathcal{A}(P),\ f \in C^\infty\big(\mathcal{A}(P)\big)$$

and similarly for β, γ. We will denote the collection of vector fields of this type by

$\mathfrak{X}_0\big(\mathcal{A}(P)\big) \leq \mathfrak{X}\big(\mathcal{A}(P)\big)$. We have

$$\begin{aligned}
\alpha_A\big[\beta(f)\big] &= \frac{\mathrm{d}}{\mathrm{d}s}\,\beta_{A+s\alpha_0}(f)\,\Big|_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t}\,f(A+s\alpha_0+t\beta_0)\,\Big|_{t=0}\Big|_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\,f(A+s\alpha_0+t\beta_0)\,\Big|_{s=0}\Big|_{t=0} \\
&= \beta_A\big[\alpha(f)\big]
\end{aligned}$$

As $\omega(\alpha,\beta) = \omega(\alpha_0,\beta_0) \in C^\infty\big(\mathcal{A}(P)\big)$ is constant, it follows by Proposition 3.10 of [42] that

$$\begin{aligned}
(\mathrm{d}\omega)(\alpha,\beta,\gamma) = \gamma(\omega(\alpha,\beta)) - \beta(\omega(\gamma,\alpha)) + \gamma(\omega(\alpha,\beta)) \\
- \omega([\alpha,\beta],\gamma) - \omega([\beta,\gamma],\alpha) - \omega([\gamma,\alpha],\beta) \\
= 0
\end{aligned}$$

Since the elements of $\mathfrak{X}_0\big(\mathcal{A}(P)\big)$ span the fibers of $T\mathcal{A}(P)$ we conclude that $\mathrm{d}\omega = 0$. $\square$

Thus ω is a symplectic form on $\Omega^1\big(\mathcal{A}(P)\big)$. Since P is trivial, we can identify the gauge group $\mathcal{G}$ with $C^\infty(M,G)$, as noted in [4]. Recall that the right action ρ of $\mathcal{G}$ on $\mathcal{A}(P)$ is given by

$$A \cdot g = A^g + \mathrm{d}g, \qquad A \in \mathcal{A}(P),\; g \in \mathcal{G}$$

where $A \mapsto A^g$ is the adjoint action and $\mathrm{d}g$ is the induced map taking values in $\mathfrak{g}$.

We will make use of the facts that the Lie algebra $\mathscr{g}$ of $\mathcal{G}$ can be naturally identified with $C^\infty(M, \mathfrak{g})$, and that the fundamental vector field of an element $Y \in \mathscr{g}$ is

$$\underline{Y}_A = \mathrm{d}_A(Y)$$

where

$$\mathscr{g} \cong \Omega^0(M, \mathfrak{g}) \xrightarrow{\ \mathrm{d}_A\ } \Omega^1(M, \mathfrak{g}) \cong T_A\mathcal{A}$$

is the exterior covariant derivative corresponding to $A \in \mathcal{A}(P)$. This mapping extends uniquely to a degree-1 derivation on $\Omega^*(M, \mathfrak{g})$. Recall, for example, from Section V of [4], that the action of the curvature form $F(A) \in \Omega^2(M, \mathfrak{g})$ is equivalently expressed by

$$F(A) \cdot Y = (\mathrm{d}_A \circ \mathrm{d}_A)(Y) = [\mathrm{d}_A A, Y], \qquad Y \in \Omega^0(M, \mathfrak{g}) \cong \mathfrak{g}$$

Finally, Proposition V.1.1. of [4] asserts that for any $\alpha_0 \in \Omega^1(M, \mathfrak{g})$, we have

$$F(A + \alpha_0) = F(A) + \mathrm{d}_A\alpha_0 + [\alpha_0, \alpha_0] \tag{$*$}$$

Proposition 10.3. *The action of $\mathcal{G}$ on $\mathcal{A}(P)$ is Hamiltonian with comoment map $\tilde{\mu} : \mathscr{g} \to C^\infty\big(\mathcal{A}(P)\big)$ defined by*

$$\tilde{\mu}(Y)(A) = \int_M Y \wedge F(A) \wedge \eta$$

for $Y \in \mathfrak{g}$ *and* $A \in \mathcal{A}(P)$.

Proof. We take this opportunity to illustrate a direct proof by means of constructing a comoment map. We will first show that $\tilde{\mu}$ is a morphism of Lie algebras, and second that it makes the above diagram commute.

Claim 1. The map $\tilde{\mu}$ is a morphism of Lie algebras.

Let $Y, Z \in \mathcal{g} \cong \Omega^0(M, \mathfrak{g})$ and $A \in \mathcal{A}(P)$ be arbitrary. By the Ad-invariance of $\langle \cdot, \cdot \rangle$, we have

$$\begin{aligned}
[Y, Z] \wedge F(A) &= \operatorname{ad}_Y Z \wedge F(A) \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}s} \operatorname{Ad}_{e^{sY}} Z \,\Big|_{s=0} \right) \wedge F(A) \\
&= \frac{\mathrm{d}}{\mathrm{d}s} \operatorname{Ad}_{e^{sY}} Z \wedge F(A) \,\Big|_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s} Z \wedge \operatorname{Ad}_{e^{-sY}} F(A) \,\Big|_{s=0} \\
&= - Z \wedge [Y, \mathrm{d}_A A] \\
&= Z \wedge (\mathrm{d}_A \circ \mathrm{d}_A) Y
\end{aligned}$$

Thus,

$$\begin{aligned}
H_{[X,Y]}(A) &= \int_M [X,Y] \wedge F(A) \wedge \eta \\
&= \int Z \wedge (\mathrm{d}_A \circ \mathrm{d}_A) Y \wedge \eta \\
&= \int \mathrm{d}_A Y \wedge \mathrm{d}_A Z \wedge \eta \;+\; \mathrm{d}(Z \wedge \mathrm{d}_A Y) \wedge \eta \\
&= \int \mathrm{d}_A Y \wedge \mathrm{d}_A Z \wedge \eta \\
&= \omega_A(\underline{Y}_A, \underline{Z}_A) \\
&= \{H_Y, H_Z\}(A)
\end{aligned}$$

Note that $\int \mathrm{d}(Z \wedge \mathrm{d}_A Y) \wedge \eta = 0$ since η is closed. This establishes Claim 1.

Claim 2. The following diagram commutes.

$$\begin{tikzcd}
 & C^\infty\big(\mathcal{A}(P)\big) \arrow[d, "f \mapsto H_f"] \\
\mathscr{g} \cong \Omega^0(M, \mathfrak{g}) \arrow[ur, "\tilde{\mu}"] \arrow[r, "Y \mapsto (\mathrm{d}_A Y)_A"'] & \mathfrak{X}\big(\mathcal{A}(P)\big)
\end{tikzcd}$$

Fix $Y \in \mathscr{g} \cong \Omega^0(M, \mathfrak{g})$ and $A \in \mathcal{A}(P)$. It remains to show,

$$\mathrm{d}_A Y = H_{\tilde{\mu}(Y)}(A)$$

We will prove an equivalent condition: that for any $\alpha \in \mathfrak{X}\big(\mathcal{A}(P)\big)$

$$\omega_A(\alpha_A, \mathrm{d}_A Y) = \alpha_A\big(\tilde{\mu}(Y)\big)$$

Let $\alpha \in \Omega^1(M, \mathfrak{g})$ so that

$$\alpha_A(f) = \frac{\mathrm{d}}{\mathrm{d}s} f(A + s\alpha_0)\Big|_{s=0}, \qquad f \in C^\infty\big(\mathcal{A}(P)\big)$$

Then,

$$\begin{aligned}
\alpha_A(\tilde{\mu}(Y)) &= \frac{\mathrm{d}}{\mathrm{d}s} \int_M Y \wedge F(A + s\alpha_0) \wedge \eta \Big|_{s=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}s} \int Y \wedge \Big(F(A) + s\, \mathrm{d}_A \alpha_0 + s^2 \,[\alpha_0, \alpha_0]\Big) \wedge \eta \Big|_{s=0}, \qquad \text{by } (*) \\
&= \int Y \wedge \mathrm{d}_A \alpha \wedge \eta \\
&= \int \alpha_A \wedge \mathrm{d}_A Y \wedge \eta + \mathrm{d}(\alpha_A \wedge Y) \wedge \eta \\
&= \int \alpha_A \wedge \mathrm{d}_A Y \wedge \eta \\
&= \omega_A(\alpha_A, \mathrm{d}_A Y)
\end{aligned}$$

This establishes Claim 2. ◻

We thus obtain the following corollary.

Corollary 10.1. *The moment map $\mu : \mathcal{A} \to \mathcal{g}^*$ of the action of $\mathcal{G}$ on $\mathcal{A}(P)$ is given* *by*

$$\mu(A)(Y) = \int_M F(A) \wedge Y \wedge \eta$$

for $A \in \mathcal{A}(P)$ and $Y \in \mathfrak{g}$. We abbreviate this as

$$\mu(A) = F(A) \wedge \eta$$

when the pairing under integration is understood.

The Marsden-Weinstein symplectic reduction theorem ensures that there is a unique reduced symplectic form ω_0 on the regular part of

$$\mathcal{A}(P)_0 = \mu^{-1}(0)/\mathcal{G}$$

which satisfies the conditions in Theorem 3.1. While the reduced space $\mathcal{A}(P)_0$ contains the moduli space $\mathcal{M}(P) = F^{-1}(0)/\mathcal{G}$ of flat connections on P, the two are not in general equal. However, it does transpire that if the symplectic form $\eta \in \Omega^2(M)$ is Lefschetz, then the regular part of $\mathcal{M}(P)$ is a symplectic submanifold of $\mathcal{A}(P)_0$. Recall that a symplectic manifold (M^{2n}, ω) is said to be *Lefschetz* when

$$H^1(M, \mathbb{R}) \to H^{2n-1}(M, \mathbb{R})$$

$$\alpha \mapsto \alpha \wedge [\omega]^{n-1}$$

is an isomorphism.

We refer to Appendix A for further details on the reduced space $\mathcal{A}(P)_0$.

10.2 Local Systems and the Space of Connections

Let M be a manifold with a symplectic structure ω (resp. Riemannian structure g). Note that ω (resp. g) induces a volume form on M and $H^1(M, \mathbb{R})$. Moreover, if G is a Lie group with an Ad-invariant inner product on the Lie algebra $\mathfrak{g}$, then there is an induced volume form on $H^1(M, \mathfrak{g})$.

For $\rho \in \mathrm{Hom}(\pi_1, G)$ and $k \in \mathbb{N}$, define the chain groups

$$C_{\rho,k}(M, \mathrm{ad}\,\mathfrak{g}) = C_k(\tilde{M}, \mathbb{Z}) \otimes_{\mathbb{Z}[\pi_1]} \mathfrak{g}$$

where $\mathbb{Z}[\pi_1]$ acts on $\tilde{M}$ by deck transformations, and on $\mathfrak{g}$ by the linear extension of

$$\gamma \cdot Y = \mathrm{Ad}_{\rho(\gamma)} Y \qquad (\gamma \in \pi_1)$$

The boundary operator

$$\partial_\rho : C_{\rho,k}(M, \mathrm{ad}\,\mathfrak{g}) \to C_{\rho,k-1}(M, \mathrm{ad}\,\mathfrak{g})$$

is induced by

$$\partial : C_k(\tilde{M}, \mathbb{Z}) \to C_{k-1}(\tilde{M}, \mathbb{Z})$$

and the identity transformation on $\mathfrak{g}$. We will denote the homology of this complex by $H_{\rho,k}(M, \mathfrak{g})$. The cohomology $H^k_\rho(M, \mathfrak{g})$ is defined similarly, with cochain groups

$$C^k_\rho(M, \operatorname{ad}\mathfrak{g}) = \operatorname{Hom}_{\mathbb{Z}[\pi]}(C_k(\tilde{M}), \mathfrak{g})$$

Refer to section 3.H of [29] for further details.

Now suppose that $P_G(M)$ is a G-principal bundle on M, let A be a flat connection on P. Let

$$\Omega^k(M, \operatorname{ad}\mathfrak{g}) = \Omega^k(M, \mathbb{R}) \otimes C^\infty(M, \operatorname{ad}\mathfrak{g})$$

be the collection of vertical G-invariant k-forms on P_G. The exterior covariant derivative d_A is a coboundary operator on the complex $(\Omega^k(M, \operatorname{ad}\mathfrak{g}), \mathrm{d}_A)$, and we denote the cohomology by $H_A(M, \operatorname{ad}\mathfrak{g})$.

If the connection A corresponds to the representation ρ, then there is a canonical identification $H^k_\rho(M, \operatorname{ad}\mathfrak{g}) \cong H^k_A(M, \operatorname{ad}\mathfrak{g})$.

As it forms the central construction of this and the following chapter, let us recall the definition of the moduli space of flat G-connections on M.

Definition. Fix a compact manifold M and a Lie group G. Let $\mathcal{P}_G(M) = \{P_i\}_i$ denote a fixed collection of G-principal bundles on M such that $\mathcal{P}_G(M)$ contains precisely one representative from each isomorphism class of G-principal bundles on M. Let $\mathcal{A}_G(M) = \bigcup_i \mathcal{A}(P_i)$ denote the union of the spaces of connections, and define the *moduli space of flat G-connections on M* to be the union $\mathcal{M}_G(M) = \bigcup_i \mathcal{M}(P_i)$ be the disjoint union of the moduli spaces $\mathcal{M}(P_i)$.

10.3 Abelian Structure Group G

Before we begin, let us first recall the definition of the

Definition. Let (M^{2n}, ω) be a symplectic manifold. The *symplectic volume* of M is the quanitity

$$\operatorname{vol} M = \frac{1}{n!} \int_M \omega^n$$

The $(2n)$-form $\frac{1}{n!}\omega^n \in \Omega^{2n}(M)$ is called the *symplectic volume form.*

We first consider the relatively straightforward case in which the structure group G is abelian. The approach developed here will later be adapted to the more sophisticated context of Section 10.4.

Theorem 10.1. *Let M be a symplectic (resp. Riemannian) manifold and let T be a compact abelian Lie group equipped with an* Ad*-invariant metric. Then*

$$\operatorname{vol} \mathcal{M}_T(M) = \operatorname{vol}(T)^{b_1(M)} \operatorname{vol} H^1(M, \mathbb{Z}) \, |\operatorname{Hom}(H_1(M, \mathbb{Z})_{Tor}, T)|$$

where $\operatorname{vol} H^1(M, \mathbb{Z})$ *denotes the lattice covolume of $H^1(M, \mathbb{Z}) \leq H^1(M, \mathbb{R})$ with re-*

spect to the symplectic (resp. Riemannian) structure on M and $\mathrm{Ch}(H_1(M,\mathbb{Z})_{Tor})$ *is the finite set of characters of* $H_1(M,\mathbb{Z})_{Tor}$*.*

Proof. Let $\mathcal{B}$ be a basis of $H_1(M,\mathbb{Z})_{\mathrm{modTor}}$. Since T is abelian, we have the following diffeomorphisms,

$$\begin{aligned}
\mathcal{M}_T(M) &\cong \mathrm{Hom}(\pi_1,T)/T \\
&\cong \mathrm{Hom}(H_1(M,\mathbb{Z}),T) \\
&\cong \mathrm{Hom}(H_1(M,\mathbb{Z})_{\mathrm{modTor}},T) \oplus \mathrm{Hom}(H_1(M,\mathbb{Z})_{\mathrm{Tor}},T) \\
&\cong C^\infty(\mathcal{B},T)\times F
\end{aligned}$$

where F is a finite set. Denote this identification by

$$\phi : \mathrm{Hom}(H_1(M,\mathbb{Z}),T) \to C^\infty(\mathcal{B},T)\times F$$

Let A be a T-connection on M and let $\rho \in \mathrm{Hom}(\pi_1,T)$ be the associated π_1-representation on T. Since T is abelian the adjoint action on $\mathfrak{t}$ is trivial, and so

$$H^1_\rho(M,\mathfrak{t}) \cong \mathrm{Hom}(H_1(M,\mathbb{Z})_{\mathrm{modTor}},\mathfrak{t})$$

Under this identification, the induced map ϕ_* is given by

$$\begin{aligned} \phi_* : \mathrm{Hom}(H_1(M,\mathbb{Z})_{\mathrm{modTor}}, \mathfrak{t}) &\longrightarrow C^\infty(\mathcal{B}, \mathfrak{t}) \\ \alpha \quad &\longmapsto \quad \alpha|_{\mathcal{B}} \end{aligned}$$

Denote by $\mathrm{dvol}_{\mathfrak{t}} \in \det \mathfrak{t}$ the metric volume form on $\mathfrak{t}$ and let $\nu \in C^\infty(\mathcal{B}, \det \mathfrak{t}) \cong \det C^\infty(\mathcal{B}, \mathfrak{t})$ be the map with constant value $\mathrm{dvol}_{\mathfrak{t}}$. Note that

$$\int_{C^\infty(\mathcal{B},T)} \nu = \mathrm{vol}(T)^{b_1(M)}$$

Let $\Lambda \leq \mathfrak{t}$ be a lattice with $\mathrm{vol}_{\mathfrak{t}}(\Lambda) = 1$ and let $\Lambda^{\mathcal{B}} = C^\infty(\mathcal{B}, \Lambda)$. Since $\phi_* H^1(M, \Lambda) = \Lambda^{\mathcal{B}}$, it follows that

$$\phi^* \nu \; H^1(M, \Lambda) = \nu(\Lambda^{\mathcal{B}}) = 1$$

From this and the equality $\mathrm{vol}(\Lambda) = 1$, we have

$$\langle \mathrm{dvol}_{\mathcal{M}_T(M)}, \nu \rangle = \mathrm{dvol}_{\mathcal{M}_T(M)}(H^1(M, \Lambda)) = \mathrm{dvol}_{H^1(M,\mathbb{R})} \, H^1(M, \mathbb{Z})$$

and we conclude that

$$\begin{aligned} \mathrm{vol}(\mathcal{M}_T(M)) &= \int_{C^\infty(\mathcal{B},T)\times F} \langle \mathrm{dvol}_{\mathcal{M}_T(M)}, \nu \rangle \, \nu \\ &= \mathrm{vol}(T)^{b_1(M)} \, \mathrm{vol} \, H^1(M, \mathbb{Z}) \, |\mathrm{Hom}(H_1(M, \mathbb{Z})_{\mathrm{Tor}}, T)| \end{aligned}$$

$\square$

Corollary 10.2. *The moduli space of complex line bundles over M with flat connection has volume*

$$\mathrm{vol}(\mathcal{M}_{U(1)}(M)) = (2\pi)^{b_1(M)} \,\mathrm{vol}\, H^1(M,\mathbb{Z})\, |\mathrm{Ch}(H_1(M,\mathbb{Z})_{Tor})|$$

where $\mathrm{Ch}(H_1(M,\mathbb{Z})_{Tor})$ is the set of characters of $H_1(M,\mathbb{Z})_{Tor}$.

Corollary 10.3. *The volume of the moduli space of Kähler metrics on the torus T^2 is*

$$\mathrm{vol}(\mathcal{M}_{U(1)}(T^2)) = 4\pi^2$$

10.4 Free Abelian Fundamental Group $\pi_1(M)$ and Semisimple Structure Group G

Let G be a compact connected semisimple Lie group. Recall that an element $g \in G$ is called *general* if g generates a maximal torus $T \leq G$.

Lemma 10.1. *Let $k \geq 1$ and define $S^{(k)} = \left\{(g_i)_i \in G^k \,\middle|\, [g_i, g_j] = 1\right\}$. Let $U^{(k)} \subseteq S^{(k)}$ consist of those elements $(g_i)_{i\leq k} \in S^{(k)}$ for which there exists a unique maximal torus $T \leq G$ with $\langle g_i \rangle_i = T$.*

(i) The set $U^{(k)}$ has full measure in $S^{(k)}$.

(ii) $U^{(k)}$ *contains a dense open subset of* $S^{(k)}$.

(iii) Let $\bar{g} = (g_i)_i \in U^{(k)}$ *and let* $\underline{\mathfrak{g}}_{\bar{g}} \leq T_{\bar{g}}S^{(k)}$ *be the induced image of* $\mathfrak{g}$ *at* $\bar{g}$ *under the action of* G *on* S *by conjugation. Then* $T_{\bar{g}}S^{(k)} \cong T_{\bar{g}}H^k \oplus \underline{\mathfrak{g}}_{\bar{g}}$ *with respect to the Killing metric.*

Proof. (i) When $k = 1$, $U^{(1)}$ is the collection of general elements in G, and $U^{(1)}$ has full measure in $G = S^{(1)}$ by Theorem 2.11 of [7]. Now suppose the claim is true for $k \geq 1$ with $U^{(k)} \subseteq S^{(k)}$ the corresponding set of full measure, and let

$$p_k : S^{(k+1)} \to S^{(k)}$$

be the projection onto the first k coordinates. Choose $(g_i)_i \in U^{(k)}$ and let $T \leq G$ be the corresponding maximal torus. Observe that

$$(g_1, \ldots, g_k, h) \in S^{(k+1)}$$

precisely when $h \in T$, and note that $\langle h \rangle = T$ for almost every $h \in T$. Thus $U^{(k+1)}$ has full measure along the fibers of $p_k|_{U^{(k)}}$. It follows that $U^{(k+1)}$ has full measure in the preimage $p_k^{-1}U^{(k)}$, which in turn has full measure in $S^{(k+1)}$. Thus for almost every $(g_i)_i \in S^{(k)}$ there is a maximal torus T which is generated by each component g_i. According to Theorem 2.11 of [7], it follows that for each g_i, T is the unique maximal torus containing g_i.

(ii) This follows from (i) and an argument from smoothness.

(iii) Since $H^k \subseteq S^{(k)}$ we have $T_{\bar{g}} H^k \leq T_{\bar{g}} S^{(k)}$. Now suppose that $(X_i)_i \in (T_{\bar{g}} H^k)^{\perp} \leq T_{\bar{g}} S^{(k)}$. Let Φ contain the roots of $\mathfrak{g} = T_1 G$ with respect to H, and note that the root-space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$$

implies that for each $i = 1, \ldots, k$, there is a $Y_i \in \mathfrak{g}$ such that $X_i = \bar{Y}_{i,g_i}$. Since any variation of $\bar{g}$ induces a variation of H, it follows that $Y_i = Y_j$ for each $i, j \leq k$. Thus $X = \bar{Y}_{i,\bar{g}}$ and we conclude that $(T_{\bar{g}} H)^{\perp} = \underline{\mathfrak{g}}_{\bar{g}}$.

□

Theorem 10.2. *Let M be a symplectic (resp. Riemannian) manifold with free abelian fundamental group $\pi_1 M$, G a compact connected semisimple Lie group of dimension k and rank ℓ, $\langle \, , \rangle$ an* Ad*-invariant metric on the Lie algebra $\mathfrak{g}$, H a maximal torus of G with Lie algebra $\mathfrak{h}$, W the Weyl group, $\{\alpha\} \subseteq H^*$ the root system, and $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ the half sum of a subsystem of positive roots. Then*

$$\operatorname{vol} \mathcal{M}_G(M) = \Big(\frac{\operatorname{vol} G}{\sqrt{2\pi}^{k-\ell}} \prod_{\alpha > 0} \alpha\rho \Big)^{b_1(M)} \frac{1}{|W|} \operatorname{vol} H^1(M, \mathbb{Z})$$

where $\operatorname{vol} H^1(M, \mathbb{Z})$ *denotes the covolume the lattice $H^1(M, \mathbb{Z})$ in $H^1(M, \mathbb{R})$.*

Proof. As we are only interested in the metric volume form $\operatorname{dvol}_{\langle , \rangle}$ on G, we will assume, without loss of meaningful generality, that $\langle\, ,\rangle$ is proportional to the Killing metric.

Let $\mathcal{B}$ be a basis for $H_1(M,\mathbb{Z})$ and let

$$C_0^\infty(\mathcal{B},G) = \big\{ f \in C^\infty(\mathcal{B},G) \,\big|\, [f(\mathcal{B}), f(\mathcal{B})] = 0 \big\}$$

so that

$$\begin{aligned} \mathrm{Hom}(\pi_1 M, G)/G &\cong \mathrm{Hom}(H_1(M,\mathbb{Z}), G)/G \\ &\cong C_0^\infty(\mathcal{B},G)/G \end{aligned}$$

Let $U_0(\mathcal{B},G) \subseteq C_0^\infty(\mathcal{B},G)$ contain precisely the elements $f : \mathcal{B} \to G$ for which there is a maximal torus $T_f \leq G$ such that, for each $\sigma \in \mathcal{B}$, T_f is the unique maximal torus containing $f(\sigma)$. Fix $f \in U_0(\mathcal{B},G)$, let $H = \big\langle f(\mathcal{B}) \big\rangle \leq G$ be the maximal torus generated by $f(\mathcal{B})$, and let $\mathfrak{h} \leq \mathfrak{g}$ be the Cartan subalgebra corresponding to H.

Denote by $\underline{\mathfrak{g}}_f \leq T_f\, C_0^\infty(\mathcal{B},G)$ the action-induced image of $\mathfrak{g}$ at f. It follows from Lemma 10.1 that

$$T_f\, C_0^\infty(\mathcal{B},G) = T_f C^\infty(\mathcal{B},H) \oplus \underline{\mathfrak{g}}_f$$

Thus, for $\bar{f} = [f]_G$,

$$\begin{aligned} T_{\bar{f}}\left(C_0^\infty(\mathcal{B}, G)/G\right) &\cong T_f C^\infty(\mathcal{B}, H) \\ &\cong C^\infty(\mathcal{B}, \mathfrak{h}) \end{aligned}$$

Let $\mathrm{dvol}_{\mathfrak{h}} \in \det \mathfrak{h}$ be the volume form induced by the Killing metric and let $\nu_f \in \det C^\infty(\mathcal{B}, \mathfrak{h}) \cong C^\infty(\mathcal{B}, \det \mathfrak{h})$ correspond to the function with constant value $\mathrm{dvol}_{\mathfrak{h}}$.

Let

$$U_H(\mathcal{B}, G) = \{f \in U_0(\mathcal{B}, G) \mid \langle f(\mathcal{B})\rangle = H\}$$

As the action of G identifies all maximal tori, there is an isometry

$$U_0(\mathcal{B}, G)/G \cong U_H(\mathcal{B}, G)/W_H$$

where we recall that W_H is the Weyl group of H. Since Lemma 10.1 provides that $U_H(\mathcal{B}, G) \subseteq H^{b_1(M)}$ has full measure, and since W_H acts by isometries, we have

$$\mathrm{vol}\left(C_0^\infty(\mathcal{B}, G)/G\right) = \mathrm{vol}\big(U_H(\mathcal{B}, G)/W_H\big) = \frac{1}{|W|}\mathrm{vol}(H)^{b_1(M)}$$

Fix $f \in U_H(\mathcal{B}, G)$ and let $\rho \in \mathrm{Hom}(\pi_1, G)$ be the corresponding representation. As $\big\langle f(\mathcal{B})\big\rangle = H$, it follows that

$$(\mathrm{Ad} \circ \rho)(\pi_1) = \mathrm{Ad}(H) = 1 \in \mathrm{Aut}(\mathfrak{h})$$

and thus

$$\begin{aligned} H^1_\rho(M, \operatorname{ad}\mathfrak{h}) &\cong H^1(M, \mathfrak{h}) \\ &\cong \operatorname{Hom}(H_1(M, \mathbb{Z}), \mathfrak{h}) \\ &\cong C^\infty(\mathcal{B}, \mathfrak{h}) \end{aligned}$$

Proceeding as in the proof of Theorem 10.1, we obtain

$$\langle \mathrm{dvol}_{\mathcal{M}_G(M)}, \nu_f \rangle = \operatorname{vol} H^1(M, \mathbb{Z})$$

from which we deduce

$$\begin{aligned} \operatorname{vol} \mathcal{M}_G(M) &= \frac{1}{|W|} \int_{U_H(\mathcal{B},G)} \langle \mathrm{dvol}_{\mathcal{M}_G(M)}, \nu_f \rangle \, \nu_f \\ &= \frac{1}{|W|} \operatorname{vol}(H)^{b_1(M)} \operatorname{vol} H^1(M, \mathbb{Z}) \end{aligned}$$

The result follows from the fact, established in [54], that

$$\operatorname{vol} H = \frac{\operatorname{vol} G}{\sqrt{2\pi}^{k-\ell}} \prod_{\alpha > 0} \alpha\rho$$

$\square$

Remark 10.1. It may appear that we have claimed that the volume of the torus H scales linearly with the volume of the group G. This apparent inconsistency is accounted for by the fact that the metric $\langle\,,\rangle$ appears in the pairing $\rho\alpha$.

Remark 10.2. The effect of the requirement that $\pi_1 M$ be free abelian is to ensure that the generic representation $\rho \in \mathrm{Hom}(\pi_1 M, G)$ satisfy $\rho(\pi_1 M) \leq T_\rho$ for some maximal torus $T_\rho \leq G$. In fact, it is this latter condition which enables the computation of the volume, as it implies that we may replace $\mathrm{Hom}(\pi_1 M, G)/G$ with $\mathrm{Hom}(\pi_1 M, T)/W \cong \mathrm{Hom}\big(H_1(M), T\big)/W$. This may enable the computation of $\mathrm{vol}\,\mathcal{M}_G(M)$ for a broader class of $\pi_1 M$ and G.

Using Table 1 of [57], we obtain the following numerical formulas for volume of the moduli space with respect to the metric conventions of Chapter VII §13 of [6].

G	$\mathrm{vol}\,\mathcal{M}_G(M)$	
$\mathrm{SU}(r+1)$	$\frac{1}{(r+1)!}\big[\pi^r(2r+2)^{r(r+2)/2}(r+1)^{3/2}\big]^{b_1(M)}$	$\mathrm{vol}\,H^1(M,\mathbb{Z})$
$\mathrm{SO}(2r+1)$	$\frac{1}{2^r r!}\big[4\pi^r(4r-2)^{r(2r+1)/2}\big]^{b_1}$	$\vdots$
$\mathrm{Sp}(2r)$	$\frac{1}{2^r r!}\big[2\pi^r(4r+4)^{r(2r+1)/2}\big]^{b_1}$	
$\mathrm{SO}(2r)$	$\frac{1}{2^{r-1} r!}\big[\pi r^2(4r-4)^{r(2r-1)/2}\big]^{b_1}$	
E_6	$\frac{1}{3\cdot 4!\,6!}\big[3^{3/2}\pi^6\,24^{39}\big]^{b_1}$	
E_7	$\frac{1}{4!\,4!\,7!}\big[2^{3/2}\pi^7\,6^{133}\big]^{b_1}$	
E_8	$\frac{1}{4!\,6!\,8!}\big[\pi^8\,60^{124}\big]^{b_1}$	
F_4	$\frac{1}{2!\,4!\,4!}\big[2\pi^4\,18^{26}\big]^{b_1}$	
G_2	$\frac{1}{12}\big[3^{1/2}\pi^2\,24^{12}\big]^{b_1}$	

The corresponding volume for an arbitrary choice of Ad-invariant metric $\langle\, , \rangle$ on $\mathfrak{g}$ can be obtained by noting that $\operatorname{vol}\mathcal{M}_G(M)$ scales linearly with $\langle\, , \rangle^{b_1(M)\operatorname{rank}G/2}$. We also observe that $\operatorname{vol}\mathcal{M}_G(M)$ also scales linearly with $\operatorname{vol}(G)^{b_1(M)\operatorname{rank}G/\dim G}$.

Corollary 10.4. *If $M = T^k$ is a k-dimensional torus with Riemannian metric g, then*

$$\operatorname{vol}\mathcal{M}_G(T^k) = \Big(\frac{\operatorname{vol}G}{\sqrt{2\pi}^{k-\ell}}\prod_{\alpha>0}\alpha\rho\Big)^{b_1(M)}\frac{1}{|W|\operatorname{vol}T^k}$$

Proof. Let $A_{T^k} = H_1(T^k,\mathbb{R})/H_1(T^k,\mathbb{Z})$ be the Albanese variety of T^k. Then $A_{T^k} \cong T^k$ and $\operatorname{vol}(A_{T^k}) = \operatorname{vol}(T^k)$ with the induced volume on A_{T^k}. Thus

$$\begin{aligned}\operatorname{vol}H^1(M,\mathbb{Z}) &= \operatorname{vol}H_1(M,\mathbb{Z})^{-1}\\ &= \operatorname{vol}(A_{T^k})^{-1}\\ &= \operatorname{vol}(T^k)^{-1}\end{aligned}$$

and the result follows by Theorem 10.2. □

Remark 10.3. At the opening of this chapter, we cited the ergodicity of the action of the mapping class group of a surface Σ on the moduli space $\mathcal{M}_G(\Sigma)$ [25]. It would be interesting to adapt this argument to the context of a higher dimension base space M, either with respect to the classical symplectic volume reviewed in this chapter, or the vector-valued volume of Chapter 7. In the latter case, the notion of ergodicity would require a suitable interpretation in the setting of a vector-valued measure.

Chapter 11

Immersions of the Moduli Space

We turn now to relate the moduli spaces $\mathcal{M}_G(M)$ and $\mathcal{M}_G(\Sigma)$ when Σ is an embedded submanifold of M. This chapter follows Section 10.1, but is otherwise independent of Chapter 10. In brief, we let (M^4, η) be a symplectic manifold, G a Lie group admitting an Ad-invariant metric $\langle\, , \rangle$ on the Lie algebra $\mathfrak{g}$, P a G-principal bundle on M, $\omega(\,\cdot, \cdot\,) = \int_M \cdot \wedge \cdot \wedge \eta$ the induced classical symplectic structure on $\mathcal{A}(P)$, $\mathcal{G}$ the gauge group, $\mathcal{A}(P)_0$ the reduction of $\mathcal{A}(P)$ by the action of $\mathcal{G}$, and $\mathcal{M}(P)$ the moduli space of flat connections.

11.1 Kähler Embeddings and Volume Comparison

Recall that a Riemannian manifold (M, g) is said to be *formal* when the subspace of harmonic forms $\mathcal{H}^*(M) \leq \Omega^*(M)$ is an subalgebra under the wedge product.

Theorem 11.1. *Let (M, g, ω) be a formal Kähler manifold, let $(\Sigma, g_\Sigma, \omega_\Sigma)$ be a Kähler surface, and suppose that $i : \Sigma \hookrightarrow X$ is a Kähler embedding. Note that the Lefschetz condition implies that the induced map $i_* : H_1(M) \xrightarrow{\sim} H_1(\Sigma)$ is a bijection. Let Ω_M and Ω_Σ be the induced symplectic structures on $H^1(M)$ and $H^1(\Sigma)$, respectively. Define $\nu_\Sigma \in H^2\big(H^1(X)\big)$ to be the form given by*

$$\nu_\Sigma(\alpha, \beta) = \int_\Sigma \langle \alpha \wedge \beta, \omega_\perp \rangle$$

where $\omega_\perp$ is the fiberwise projection of ω onto the normal fibers $N_x\Sigma \subseteq T_xM|_\Sigma$ along Σ. Then

$$\frac{\mathrm{vol}(M)}{n\,\mathrm{vol}(\Sigma)}\,\Omega_M = i^*\Omega_\Sigma + \nu_\Sigma$$

Proof. Let $f_t \in C^\infty(M)$ be a solution to the heat equation

$$\frac{\mathrm{d}}{\mathrm{d}t} f_t = \Delta f_t$$

with initial data

$$\lim_{t\to 0} \int_M \phi f_t \,\mathrm{dvol}_M = \int_\Sigma \phi \,\mathrm{dvol}_\Sigma, \qquad \phi \in C^\infty(M)$$

Write $\eta = \frac{1}{(n-1)!}\omega^{n-1}$ and let $\alpha, \beta \in \mathcal{H}^1(M)$. It follows that $\alpha \wedge \beta \wedge \eta$ is harmonic, and

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_M \alpha\wedge\beta\wedge\eta\, f_t &= \int \alpha\wedge\beta\wedge\eta\,\Delta f_t \\
&= \int \big\langle *(\alpha\wedge\beta\wedge\eta), \mathrm{d}^*\mathrm{d}f_t\big\rangle\,\mathrm{dvol}_X \\
&= 0
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
\frac{\mathrm{vol}(\Sigma)}{n\,\mathrm{vol}(M)}\Omega_M(\alpha,\beta) &= \frac{\mathrm{vol}(\Sigma)}{n\,\mathrm{vol}(M)}\int_M \alpha\wedge\beta\wedge\eta \\
&= \frac{1}{n}\lim_{t\to\infty}\int_M \alpha\wedge\beta\wedge f_t\eta \\
&= \frac{1}{n}\lim_{t\to 0}\int_M \alpha\wedge\beta\wedge f_t\eta \\
&= \lim_{t\to 0}\int_M \langle\alpha\wedge\beta,\omega\rangle f_t\,\mathrm{dvol}_M \\
&= \int_\Sigma \langle\alpha\wedge\beta,\omega\rangle\,\omega \\
&= \int_\Sigma \langle\alpha\wedge\beta,\omega_\Sigma\rangle\,\omega + \int_\Sigma \langle\alpha\wedge\beta,\omega_\perp\rangle\,\omega \\
&= (i^*\Omega_\Sigma)(\alpha,\beta) + \nu_\Sigma(\alpha,\beta)
\end{aligned}
$$

□

Remark 11.1. Note that,

(i) If (M,g,ω) is not formal, then, without loss of meaningful generality, we may replace $\alpha\wedge\beta$ with the harmonic representative of its cohomology class.

(ii) The inequality

$$\|\Omega_M\| \leq \|i_*\| \, \|\Omega_\Sigma\| + \|\nu\|$$

can be used to compare the symplectic the volumes of $\mathcal{M}(P_M)$ and $\mathcal{M}(P_\Sigma)$ for $U(1)$-principal bundles P_M and P_Σ. This approach could potentially be adapted to the case of nonabelian structure groups G.

11.2 Canonical Immersions of $\mathcal{M}_G(M)$ in $\mathcal{M}_G(\Sigma)$

We first show that the moduli space construction $\mathcal{M} : P \mapsto \mathcal{M}(P)$ satisfies a property akin to that of a contravariant functor with respect to smooth and equivariant embeddings $f : P \to P'$.

Lemma 11.1. *Let G be a Lie group, let P_N and P_M be G-principal bundles on N and M, respectively, and let $f : P_N \to P_M$ be a G-equivariant smooth map. Then f pulls back connection 1-forms from M to N and, moreover, the curvature*

$$F_{f^*\alpha} = f^* F_\alpha \in \Omega^2(N, \operatorname{ad}\mathfrak{g}), \qquad \alpha \in \Omega^1(P_M, \mathfrak{g})^G$$

In particular, f pulls back flat connections to flat connections. If f is an embedding, then there is an induced map $f^* : \mathcal{M}_G(P(M)) \to \mathcal{M}_G(P(N))$.

Proof. Let $\alpha \in \Omega^1(P_M, \mathfrak{g})^G$ be a connection 1-form on P_M. Since α and f are G-equivariant, it follows that $f^*\alpha$ is also G-equivariant. Thus, for $X \in \mathfrak{g}$ and $p \in P_N$, we have

$$\begin{aligned}
(f^*\alpha)(p\cdot X) &= (f^*\alpha)\Big(\frac{\mathrm{d}}{\mathrm{d}t}\, e^{tX}\cdot p\Big|_{t=0}\Big)\\
&= \alpha\Big(\frac{\mathrm{d}}{\mathrm{d}t}\, e^{tX}\cdot f(p)\Big|_{t=0}\Big)\\
&= \alpha\big(f(p)\cdot X\big)\\
&= X
\end{aligned}$$

Hence $f^*\alpha$ is a connection 1-form on P_N. The structure equation [42, Theorem 5.2] yields

$$f^*F_\alpha = \mathrm{d}(f^*\alpha) - \frac{1}{2}[f^*\alpha, f^*\alpha] = F_{f^*\alpha}$$

and it follows that flat connections pull back to flat connections. Consider the adjoint bundles

$$\mathrm{Ad}\, P_N = P_N \times_\ell G, \qquad \mathrm{Ad}\, P_M = P_M \times_\ell G$$

and the gauge groups

$$\mathcal{G}P_N = C^\infty(N, \mathrm{Ad}\, P_N), \qquad \mathcal{G}P_M = C^\infty(M, \mathrm{Ad}\, P_M)$$

Now suppose f is an embedding and note that the induced map $f_* : \mathrm{Ad}\, P_N \to \mathrm{Ad}\, P_M$ yields

$$f^* : \mathcal{G}(P_M) \to \mathcal{G}(P_N)$$

which is the composition of

$$C^\infty(M, \mathrm{Ad}\, P_M) \longrightarrow C^\infty(N, \mathrm{Ad}\, P_N) \longrightarrow C^\infty(N, \mathrm{Ad}\, P_N)$$

where the first factor is the pullback by $\bar{f} : N \to M$ and the second is induced by $f_*^{-1} : \mathrm{Ad}P_N|_{\mathrm{im} f_*} \to \mathrm{Ad}P_N$. For $x \in N$, $u \in (P_N)_x$ and $g_x \in \mathrm{Ad}_x P_N$, we have

$$f(ug_x) = f(u) \cdot f_* g_x$$

Thus, for any $g \in \mathcal{G}(P_N)$ and any $X_u \in T_u P$,

$$f_*(X_u g) = (f_* X_u) \cdot f_* g$$

Therefore, for any connection $A \leq TP$,

$$f_*(Ag) = (f_* A) \cdot f_* g$$

Dually, for any connection 1-form $\alpha \in \Omega^1(P_M, \mathfrak{g})$ and any $g \in \mathcal{G}(P_M)$, we have

$$f^*(\alpha g) = (f^* \alpha) \cdot f^* g$$

and so f^* preserves gauge-equivalence classes. $\square$

We will apply Lemma 11.1 to the case where $N = \Sigma \subseteq M$ is an embedded surface and $f : P|_\Sigma \hookrightarrow P$ is the inclusion, for a G-principal bundle P on M.

Theorem 11.2. *Suppose that (M^{2n}, ω) is a symplectic manifold and that $\frac{1}{(n-1)!}\omega \in H^{2n-2}(M, \mathbb{Z})$ is indivisible. If $n \geq 2$, then there is a compact, connected embedded surface $\Sigma \subseteq M$ such that $[\Sigma] \in H^2(M)$ is the Poincaré dual of $[\eta] \in H^{2n-2}(M)$. Moreover, the inclusion $i : \Sigma \hookrightarrow M$ yields a symplectic immersion $i^* : \mathcal{M}_G(M) \to \mathcal{M}_G(\Sigma)$ and, at a point $[A] \in \mathcal{M}_G(M)$, the codimension of the image is equal to*

$$\dim \ker\big(H^2_A(M, \Sigma; \mathrm{ad}\,\mathfrak{g}) \longrightarrow H^2_A(M, \mathrm{ad}\,\mathfrak{g}) \big)$$

Proof. Put $\eta = \frac{1}{(n-1)!}\omega^{n-1} \in H^{2n-2}(M)$. The existence of Σ follows from the indivisibility of $\eta \in H^{n-2}(M, \mathbb{Z})$ by [89]. Fix a connection A on M and let $\alpha, \beta \in H^1_A(M, \mathrm{ad}\,\mathfrak{g}) \cong T_A\mathcal{M}_G(M)$. Since Σ represents the Poincare dual of η, it follows that

$$\int_M \alpha \wedge \beta \wedge \eta = \int_\Sigma \alpha \wedge \beta$$

and so the derivative $\mathrm{d}(i^*)_A : T_A\mathcal{M}_G(N) \to T_{i^*A}\mathcal{M}_G(M)$ is symplectic and hence injective. Note that $\mathrm{d}(i^*)_A$ forms a part of the long exact sequence on cohomology induced by an inclusion,

$$
\begin{tikzcd}
H^2_A(M,\Sigma;\operatorname{ad}\mathfrak{g}) \arrow[r, "j"] & H^2_A(M,\operatorname{ad}\mathfrak{g}) \arrow[r] & H^2_A(\Sigma,\operatorname{ad}\mathfrak{g}) \\
H^1_A(M,\Sigma;\operatorname{ad}\mathfrak{g}) \arrow[r, "0"'] & H^1_A(M,\operatorname{ad}\mathfrak{g}) \arrow[r, "\mathrm{d}(i^*)_A"'] & H^1_A(\Sigma,\operatorname{ad}\mathfrak{g}) \arrow[ull, "\mathrm{d}_A"']
\end{tikzcd}
$$

The initial map is 0 since $\mathrm{d}(i^*)_A$ is injective. The proof is complete by observing that

$$
\begin{aligned}
\operatorname{corank} i^* &= \dim H^1_A(\Sigma,\operatorname{ad}\mathfrak{g}) - \operatorname{rank}\mathrm{d}(i^*)_A \\
&= \dim H^1_A(\Sigma,\operatorname{ad}\mathfrak{g}) - \operatorname{null}\mathrm{d}_A \\
&= \operatorname{rank}\mathrm{d}_A \\
&= \operatorname{null} j
\end{aligned}
$$

□

Remark 11.2. Observe that the immersion i^* is invariant under local deformations of Σ. It would be interesting to determine the extent to which i^* depends on the particular choice of representative $\Sigma \subseteq M$ of the Poincaré dual $[\eta]^* \in H_2(M)$.

Theorem 11.2 implies that the unknown spaces $\mathcal{M}_G(M)$ can be understood in terms of the familiar spaces $\mathcal{M}_\Sigma(M)$. For example, if $\mathcal{M}_G(M)$ is closed, then the possible values of $\operatorname{vol}\mathcal{M}_G(M)$ are restricted to the pairings of the class $\frac{1}{k!}[\omega_{\mathcal{M}_G(\Sigma)}] \in H^*\big(\mathcal{M}_G(\Sigma)\big)$, $k \in \mathbb{N}$, with the homology classes on $\mathcal{M}_G(\Sigma)$.

Another consequence of Theorem 11.2 is that if $\mathcal{M}_G(\Sigma)$ admits a prequantum line bundle $L_{\mathcal{M}_G(\Sigma)}$, then $\mathcal{M}_G(M)$ admits a prequantum line bundle $L_{\mathcal{M}_G(M)}$ as well. For example, we may take $L_{\mathcal{M}_G(M)}$ to be the pullback of $L_{\mathcal{M}_G(\Sigma)}$ by the immersion of $\mathcal{M}_G(M)$ in $\mathcal{M}_G(\Sigma)$.

Note that the conditions of Theorem 11.2 did not require that (M, ω) satisfy the Lefschetz condition, that is, that the assignment

$$H^1(M, \mathbb{R}) \to H^{2n-1}(M, \mathbb{R})$$

$$\alpha \mapsto \alpha \wedge [\omega]^{n-1}$$

be an isomorphism, and hence that $\mathcal{M}_G(M)$ inherits a natural symplectic structure. In fact, it turns out that this condition is in fact already ensured.

Proposition 11.1. *Any symplectic manifold (M^{2n}, ω) that satisfies the hypotheses of Theorem 11.2 is Lefschetz.*

Proof. Let $\Sigma \subseteq M$ be as in Theorem 11.2. By the nondegeneracy of the symplectic form,

$$(\alpha, \beta) \mapsto [\Sigma] \cap (\alpha \cup \beta) = [\Sigma] \cap \alpha \cap \beta, \qquad \alpha, \beta \in H^1(M, \mathbb{R})$$

Consequently the

$$[\Sigma]\cap : H^1(M, \mathbb{R}) \to H_1(M, \mathbb{R})$$

$$\alpha \mapsto [\Sigma] \cap \alpha$$

is injective, and hence an isomorphism. Since the Poincaré duality map $[M]\cap$: $H^{2n-1}(M,\mathbb{R}) \to H_1(M,\mathbb{R})$ is an isomorphism, and since

$$[M] \cap (\eta \cup \alpha) = [\Sigma] \cap \alpha$$

we deduce that $\alpha \mapsto \eta \cup \alpha$ is an isomorphism. □

Appendix A

The Reduced Space $\mathcal{A}(P)_0$

The purpose of this appendix is to investigate the reduction of the space of connections by the action of the gauge group, with respect to the symplectic structure developed in Section 10.1.

Let us recall the relevant data. Let (M^4, η) be a symplectic manifold, G a Lie group admitting an Ad-invariant metric $\langle\, , \rangle$ on the Lie algebra $\mathfrak{g}$, P a G-principal bundle on M, $\omega(\,\cdot, \cdot\,) = \int_M \cdot \wedge \cdot \wedge \eta$ the induced classical symplectic structure on $\mathcal{A}(P)$, $\mathcal{G}$ the gauge group, $\mathcal{A}(P)_0$ the reduction of $\mathcal{A}(P)$ by the action of $\mathcal{G}$, and $\mathcal{M}(P)$ the moduli space of flat connections. See Section 10.1 for further details.

A.1 The Inclusion $\mathcal{M}(P) \subseteq \mathcal{A}(P)_0$

We first substantiate our claim, asserted in Section 10.1, that the reduced space $\mathcal{A}(P)_0$ contains $\mathcal{M}(P)$.

Proposition A.1. *There is an inclusion*

$$i : \mathcal{M}(P) \hookrightarrow \mathcal{A}(P)_0$$

of the moduli space of flat connections $\mathcal{M}(P)$ into the reduced space $\mathcal{A}(P)_0$.

Proof. Let $[A] \in \bar{A}$, where $[\,\cdot\,]$ represents the residue class modulo the action of $\mathcal{G}$. Then

$$\mu(A) = F(A) \wedge \eta = 0$$

so that $A \in \mu^{-1}(0)$ and thus $[A] \in \mathcal{A}(P)_0$. Consequently, $\bar{A} \subseteq \mathcal{A}(P)_0$. $\square$

The next lemma will be useful for investigating the tangent bundle $T\mathcal{A}(P)_0$.

Lemma A.1. *The map*

$$\mu_* : T\mathcal{A}(P) \to T\mathscr{g}^* \cong \mathscr{g}^*$$

is given by

$$\mu_*(\alpha_A)(Y) = \int_M \mathrm{d}_A\alpha \wedge Y \wedge \eta$$

for $A \in \mathcal{A}(P)$ and $Y \in \Omega^0(M, \mathfrak{g}) \cong \mathscr{g}$.

Proof. From Section III.1 of [4], we know that

$$\mu_*(\alpha_A)(Y) = \omega(\alpha_A, \underline{Y}_A)$$

Hence,

$$\begin{aligned}
\mu_*(\alpha_A)(Y) &= \int_M \alpha_A \wedge \underline{Y}_A \wedge \eta \\
&= \int_M \alpha_A \wedge \mathrm{d}_A Y \wedge \eta \\
&= \int_M \mathrm{d}_A \alpha_A \wedge Y \wedge \eta
\end{aligned}$$

as required. $\square$

Proposition A.2. *For $[A] \in \mathcal{A}(P)_0$, we have*

$$T_{[A]}\mathcal{A}(P) \cong H^1(\mathcal{C}_A)$$

where $\mathcal{C}_A$ is the complex

$$\Omega^0(M, \mathfrak{g}) \xrightarrow{\mathrm{d}_A} \Omega^1(M, \mathfrak{g}) \xrightarrow{\eta^\wedge \circ \mathrm{d}_A} \Omega^4(M, \mathfrak{g})$$

and where the map $\eta^\wedge : \Omega^*(M, \mathfrak{g}) \to \Omega^*(M, \mathfrak{g})$ is defined by

$$\eta^\wedge(\alpha) = \eta \wedge \alpha, \qquad \alpha \in \Omega^*(M, \mathfrak{g})$$

Proof. The tangent space of $\mu^{-1}(0)$ at A consists of all $\alpha_A \in T_A\mathcal{A}(P)$ for which $\mu_*(\alpha_A) = 0$. That is,

$$T_A\, \mu^{-1}(0) = (\mu_*)_A^{-1}(0)$$

It follows that the tangent space of $\mu^{-1}(0)/\mathcal{G}$ at $[A]$ is isomorphic to

$$(\mu_*)_A^{-1}(0)/\underline{\mathcal{g}}_{\mathrm{A}}$$

where

$$\underline{\mathcal{g}}_{\mathrm{A}} = \{\underline{Y}_A\}_{Y\in\mathcal{g}}$$

is the space of action-induced vectors at A. By Lemma A.1, we have

$$(\mu_*)_A^{-1}(0) = \ker\left(\eta^\wedge \circ \mathrm{d}_A\right)$$

The result follows as

$$\underline{\mathcal{g}}_{\mathrm{A}} = \{\mathrm{d}_A Y\}_{Y\in\mathcal{g}} = \operatorname{im} \mathrm{d}_A$$

□

Now suppose that $A \in F^{-1}(0)$. An analogous argument to that of Proposition A.2 establishes that

$$T_{[A]}\mathcal{M}(P) \cong \bar{\mathcal{C}}_A$$

where $\bar{\mathcal{C}}_A$ is the complex

$$\Omega^0(M,\mathfrak{g}) \xrightarrow{\mathrm{d}_A} \Omega^1(M,\mathfrak{g}) \xrightarrow{\mathrm{d}_A} \Omega^2(M,\mathfrak{g})$$

This is similar to the complex $\mathcal{C}_a$ in [52] Section 4. The complexes $\mathcal{C}_A$ and $\bar{\mathcal{C}}_A$ are related by the chain map,

$$\begin{tikzcd}
\bar{\mathcal{C}}_A \arrow[d] & \Omega^0(M,\mathfrak{g}) \arrow[r, "\mathrm{d}_A"] \arrow[d, "1"] & \Omega^1(M,\mathfrak{g}) \arrow[r, "\mathrm{d}_A"] \arrow[d, "1"] & \Omega^2(M,\mathfrak{g}) \arrow[d, "\eta^\wedge"] \\
\mathcal{C}_A & \Omega^0(M,\mathfrak{g}) \arrow[r, "\mathrm{d}_A"] & \Omega^1(M,\mathfrak{g}) \arrow[r, "\eta^\wedge \circ \mathrm{d}_A"] & \Omega^4(M,\mathfrak{g})
\end{tikzcd}$$

As the first two arrows are identies, the induced map

$$H^1(\bar{\mathcal{C}}_A) \hookrightarrow H^1(\mathcal{C}_A)$$

is injective and, in particular, is the derivative of the inclusion $i : \mathcal{M}(P) \to \mathcal{A}(P)_0$ from Proposition A.1,

$$(i_*)_{\bar{A}} : T_{\bar{A}}\mathcal{M}(P) \hookrightarrow T_{\bar{A}}\mathcal{A}(P)_0$$

A.2 The Dimension of $\mathcal{A}(P)_0$

We turn now to consider the dimension of $\mathcal{A}(P)_0$. It is well-known that $\dim \mathcal{M}(P)$ is finite-dimensional. Our aim is to show that $\dim \mathcal{A}(P)_0$ is finite-dimensional as well. This is not of particular importance to the main lines of inquiry taken in this dissertation, though it would be interesting to understand

1. how the manifold structure of $\mathcal{M}(P)$ relates to that of $\mathcal{A}(P)_0$, and

2. the dependence of $\mathcal{A}(P)_0$ on the choice of $\eta \in \Omega^2(M)$,

when $\dim \mathcal{A}(P)_0$ is finite-dimensional.

As a matter of notation, all chain complexes should be understood to be extended by zero on either side.

Perhaps the most straightforward route to determine $\dim \mathcal{A}(P)_0$ would be to extend the sequence $\bar{\mathcal{C}}_A \to \mathcal{C}_A$ ($A \in F^{-1}(0)$) to a short exact sequence of complexes,

$$\mathcal{B} \to \bar{\mathcal{C}}_A \to \mathcal{C}_A, \qquad \text{or} \qquad \bar{\mathcal{C}}_A \to \mathcal{C}_A \to \mathcal{B}$$

for some complex $\mathcal{B}$, and then to use the associated long exact sequence on cohomology to determine the dimension of $H^1(\mathcal{C}_A) \cong T_{[A]}\mathcal{A}(P)_0$. However, since $\eta^\wedge$ is not surjective (resp. injective), the sequence $\bar{\mathcal{C}}_A \to \mathcal{C}_A$ cannot extend on the left (resp. right). Therefore, this abstract homological approach is not available to us.

We will instead attempt a more direct proof based on the Hodge theory of the complex $\mathcal{C}_A$. Our primary reference in this section is Chapter IV, Sections 2 and 5, of [96].

Definition A.1. Let N be a manifold. A *differential complex* $\mathcal{E}$ is a chain complex

$$\Gamma(E_1) \xrightarrow{\quad L_1 \quad} \Gamma(E_2) \xrightarrow{\quad L_2 \quad} \cdots \xrightarrow{\quad L_3 \quad} \Gamma(E_n)$$

where each E_i is a vector bundle over N, and where each map

$$L_i : \Gamma(E_i) \longrightarrow \Gamma(E_{i+1})$$

is a differential operator. Let $\pi : T^*N \to N$ be the cotangent-bundle projection map.

The complex $\mathcal{E}$ is called *elliptic* if the corresponding symbol sequence

$$\pi^*(E_1) \xrightarrow{\sigma(L_1)} \pi^*(E_2) \xrightarrow{\sigma(L_2)} \cdots \xrightarrow{\sigma(L_3)} \pi^*(E_n)$$

is an exact sequence.

Example A.1. The complex $\mathcal{C}_A$ is a differential complex:

$$\Gamma(\Lambda^0 M \otimes \mathfrak{g}) \xrightarrow{\mathrm{d}_A} \Gamma(\Lambda^1 M \otimes \mathfrak{g}) \xrightarrow{\eta^\wedge \circ \mathrm{d}_A} \Gamma(\Lambda^4 M \otimes \mathfrak{g})$$

Under the assumption that $\mathcal{C}_A$ is elliptic, a standard argument for $\dim \mathcal{A}_0 < \infty$ would proceed as follows.

1. Ellipticity implies that the space of $\mathcal{C}_A$-harmonic sections is finite dimensional.

2. The space of $\mathcal{C}_A$-harmonic sections on $\Omega^1(M, \mathfrak{g})$ is isomorphic to $H^1(\mathcal{C}_A) \cong T_{[A]}\mathcal{A}$.

3. Steps 1 and 2 yield $\dim T_{[A]}\mathcal{A} < \infty$.

4. Thus $\mathcal{A}_0$ is finite-dimensional by the arbitrariness of $A \in \mu^{-1}(0)$.

However,

Proposition A.3. *The complex $\mathcal{C}_A$ is not elliptic for any $A \in \mu^{-1}(0)$.*

We require the following two lemmas.

Lemma A.2. *Fix $i \in \mathbb{N}$ and consider the maps*

$$\mathrm{d}_A,\, \eta^\wedge : \Gamma(\Lambda^i M \otimes \mathfrak{g}) \to \Gamma(\Lambda^{i+1} M \otimes \mathfrak{g})$$

For any $x \in M$ *and* $\alpha \in T_x^*M\backslash\{0\}$,

1. $\sigma(\mathrm{d}_A)_\alpha = \alpha^\wedge$

2. $\sigma(\eta^\wedge)_\alpha = \eta^\wedge$

Proof. Choose $f \in C^\infty(M)$ such that $f(x) = 0$ and $(\mathrm{d}f)_x = \alpha$, and let $\theta \in \Omega^*(M, \mathfrak{g})$ be arbitrary. For the first-order operator d_A, we have

$$\begin{aligned}
\sigma(\mathrm{d}_A)_\alpha\, \theta_x &= \big[\mathrm{d}_A(f \cdot \theta)\big]_x \\
&= \big[(\mathrm{d}f) \wedge \theta + f \cdot (\mathrm{d}_A \theta)\big]_x \\
&= \alpha \wedge \theta_x
\end{aligned}$$

For the zeroth-order operator $\eta^\wedge$,

$$\begin{aligned}
\sigma(\eta^\wedge)_\alpha\, \theta_x &= \big[\eta^\wedge(f^0 \cdot \theta)\big]_x \\
&= \eta \wedge \theta_x
\end{aligned}$$

This proves the claim. $\square$

Lemma A.3. *Fix* $x \in M$ *and* $\alpha \in \Lambda^1_x M\backslash\{0\}$*. Consider the maps*

$$\alpha^\wedge : \Lambda^0_x M \otimes \mathfrak{g} \to \Lambda^1_x M \otimes \mathfrak{g}$$

and

$$(\eta \wedge \alpha)^\wedge : \Lambda^1_x M \otimes \mathfrak{g} \to \Lambda^4_x M \otimes \mathfrak{g}$$

We have

$$\dim\left(\ker(\eta \wedge \alpha)^\wedge\right) > \dim\left(\operatorname{im}\alpha^\wedge\right)$$

Proof. Put $k = \dim \mathfrak{g}$. Since $\alpha \neq 0$, the assignment

$$\mathfrak{g} \to \Lambda^1_x M \otimes \mathfrak{g}$$
$$Y \mapsto \alpha \otimes Y$$

is injective and hence

$$\operatorname{rank}\alpha^\wedge = \dim(\Lambda^0_x M \otimes \mathfrak{g}) - \operatorname{null}\alpha^\wedge = k$$

Since $\dim(\Lambda^k_x M \otimes \mathfrak{g}) = k$, it follows that $\operatorname{rank}(\eta \wedge \alpha)^\wedge \leq k$. Consequently

$$\operatorname{null}(\eta \wedge \alpha)^\wedge = \dim(\Lambda^1 M \otimes \mathfrak{g}) - \operatorname{rank}(\eta \wedge \alpha)^\wedge \geq 4k - k$$

This completes the proof. $\square$

We are now ready to prove Proposition A.3.

Proof of Proposition A.3. We will show that the sequence of symbols

$$\pi^*(\Lambda^0 M \otimes \mathfrak{g}) \xrightarrow{\sigma(\mathrm{d}_A)} \pi^*(\Lambda^1 M \otimes \mathfrak{g}) \xrightarrow{\sigma(\eta^\wedge \circ \mathrm{d}_A)} \pi^*(\Lambda^4 M \otimes \mathfrak{g})$$

is not exact. Let $x \in M$ and $\alpha \in T_x^* M \backslash \{0\}$. By Lemma A.2 and the composition rule for symbols, we have

$$\ker\, \sigma(\eta^\wedge \circ \mathrm{d}_A)_\alpha = \ker\, \sigma(\eta^\wedge)_\alpha \circ \sigma(\mathrm{d}_A)_\alpha = \ker\, (\eta \wedge \alpha)^\wedge$$

and

$$\mathrm{im}\, \sigma(\mathrm{d}_A)_\alpha = \mathrm{im}\, \alpha^\wedge$$

Thus, it suffices to show

$$\frac{\ker\, (\eta \wedge \alpha)^\wedge}{\mathrm{im}\, \alpha^\wedge} \neq 0$$

which is an immediately consequence of Lemma A.3. □

Thus, if it is indeed the case that the reduced space $\mathcal{A}(P)_0$ is finite-dimensional, then a more sophisticated argument is required to show this.

Bibliography

[1] Anton Alekseev, Anton Malkin, and Eckhard Meinrenken. Lie group valued moment maps. *J. Differential Geom.*, 48(3):445–495, 1998.

[2] M. F. Atiyah and R. Bott. The Yang-Mills equations over Riemann surfaces. *Philos. Trans. Roy. Soc. London Ser. A*, 308(1505):523–615, 1983.

[3] Michael Atiyah. Topological quantum field theories. *Inst. Hautes Études Sci. Publ. Math.*, (68):175–186 (1989), 1988.

[4] Michèle Audin. *Torus actions on symplectic manifolds*, volume 93 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, revised edition, 2004.

[5] Raoul Bott and Loring W. Tu. *Differential forms in algebraic topology*, volume 82 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[6] Nicolas Bourbaki. *Lie groups and Lie algebras. Chapters 7–9.* Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. Translated from the 1975

and 1982 French originals by Andrew Pressley.

[7] Theodor Bröcker and Tammo tom Dieck. *Representations of compact Lie groups*, volume 98 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. Translated from the German manuscript, Corrected reprint of the 1985 translation.

[8] Ana Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[9] Beniamino Cappelletti-Montano, Antonio De Nicola, and Ivan Yudin. A survey on cosymplectic geometry. *Rev. Math. Phys.*, 25(10):1343002, 55, 2013.

[10] Paolo Casati, Franco Magri, and Marco Pedroni. The bi-Hamiltonian approach to integrable systems. In *Modern group analysis: advanced analytical and computational methods in mathematical physics (Acireale, 1992)*, pages 101–110. Kluwer Acad. Publ., Dordrecht, 1993.

[11] Claude Chevalley and Samuel Eilenberg. Cohomology theory of Lie groups and Lie algebras. *Trans. Amer. Math. Soc.*, 63:85–124, 1948.

[12] M. Condevaux, P. Dazord, and P. Molino. Géométrie du moment. In *Travaux du Séminaire Sud-Rhodanien de Géométrie, I*, volume 88 of *Publ. Dép. Math. Nouvelle Sér. B*, pages 131–160. Univ. Claude-Bernard, Lyon, 1988.

[13] Theodore James Courant. Dirac manifolds. *Trans. Amer. Math. Soc.*, 319(2):631–661, 1990.

[14] Manfredo Perdigão do Carmo. *Riemannian geometry*. Mathematics: Theory & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[15] S. K. Donaldson. Self-dual connections and the topology of smooth 4-manifolds. *Bull. Amer. Math. Soc. (N.S.)*, 8(1):81–83, 1983.

[16] S. K. Donaldson. Gluing techniques in the cohomology of moduli spaces. In *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, pages 137–170. Publish or Perish, Houston, TX, 1993.

[17] S. K. Donaldson. Remarks on gauge theory, complex geometry and 4-manifold topology. In *Fields Medallists' lectures*, volume 5 of *World Sci. Ser. 20th Century Math.*, pages 384–403. World Sci. Publ., River Edge, NJ, 1997.

[18] S. K. Donaldson and P. B. Kronheimer. *The geometry of four-manifolds*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1990. Oxford Science Publications.

[19] S. K. Donaldson and R. P. Thomas. Gauge theory in higher dimensions. In *The geometric universe (Oxford, 1996)*, pages 31–47. Oxford Univ. Press, Oxford, 1998.

[20] Simon Donaldson and Ed Segal. Gauge theory in higher dimensions, II. In *Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics*, volume 16 of *Surv. Differ. Geom.*, pages 1–41. Int. Press, Somerville, MA, 2011.

[21] René Dugas. *A history of mechanics*. Dover Publications, Inc., New York, 1988. With a foreword by Louis de Broglie, Translated from the French by J. R. Maddox, Reprint of the 1957 translation.

[22] Birte Feix. Hyperkähler metrics on cotangent bundles. *J. Reine Angew. Math.*, 532:33–46, 2001.

[23] Rui L. Fernandes. Completely integrable bi-Hamiltonian systems. *J. Dynam. Differential Equations*, 6(1):53–69, 1994.

[24] William M. Goldman. The symplectic nature of fundamental groups of surfaces. *Adv. in Math.*, 54(2):200–225, 1984.

[25] William M. Goldman. Ergodic theory on moduli spaces. *Ann. of Math. (2)*, 146(3):475–507, 1997.

[26] Hubert Goldschmidt and Shlomo Sternberg. The Hamilton-Cartan formalism in the calculus of variations. *Ann. Inst. Fourier (Grenoble)*, 23(1):203–267, 1973.

[27] Brian Hall. *Lie groups, Lie algebras, and representations*, volume 222 of *Graduate Texts in Mathematics*. Springer, Cham, second edition, 2015. An elementary introduction.

[28] William Rowan Hamilton. First essay on a general method in dynamics. *Philos. Trans. Roy. Soc.*, II:247, 1834.

[29] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[30] Thomas Hawkins. *Emergence of the theory of Lie groups*. Sources and Studies in the History of Mathematics and Physical Sciences. Springer-Verlag, New York, 2000. An essay in the history of mathematics 1869–1926.

[31] Frédéric Hélein. Multisymplectic formalism and the covariant phase space. In *Variational problems in differential geometry*, volume 394 of *London Math. Soc. Lecture Note Ser.*, pages 94–126. Cambridge Univ. Press, Cambridge, 2012.

[32] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Ro ˇ cek. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108(4):535–589, 1987.

[33] Nigel Hitchin. Hyper-Kähler manifolds. *Astérisque*, (206):Exp. No. 748, 3, 137–166, 1992. Séminaire Bourbaki, Vol. 1991/92.

[34] Nan-Kuo Ho and Lisa C. Jeffrey. The volume of the moduli space of flat connections on a nonorientable 2-manifold. *Comm. Math. Phys.*, 256(3):539–564, 2005.

[35] Johannes Huebschmann. Symplectic and Poisson structures of certain moduli spaces. I. *Duke Math. J.*, 80(3):737–756, 1995.

[36] Daniel Huybrechts. *Lectures on K3 surfaces*, volume 158 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2016.

[37] Lisa C. Jeffrey. Extended moduli spaces of flat connections on Riemann surfaces. *Math. Ann.*, 298(4):667–692, 1994.

[38] Lisa C. Jeffrey, Daniel A. Ramras, and Jonathan Weitsman. The prequantum line bundle on the moduli space of flat $SU(N)$ connections on a Riemann surface and the homotopy of the large N limit. *Lett. Math. Phys.*, 107(9):1581–1589, 2017.

[39] Jerzy Kijowski. A finite-dimensional canonical formalism in the classical field theory. *Comm. Math. Phys.*, 30:99–128, 1973.

[40] A. A. Kirillov. *Elements of the theory of representations*. Springer-Verlag, Berlin-New York, 1976. Translated from the Russian by Edwin Hewitt, Grundlehren der Mathematischen Wissenschaften, Band 220.

[41] A. A. Kirillov. *Lectures on the orbit method*, volume 64 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2004.

[42] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication.

[43] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. II*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1996. Reprint of the 1969 original, A Wiley-Interscience Publication.

[44] Bertram Kostant. Orbits, symplectic structures and representation theory. In *Proc. U.S.-Japan Seminar in Differential Geometry (Kyoto, 1965)*, page p. 71. Nippon Hyoronsha, Tokyo, 1966.

[45] J. L. Lagrange. *Analytical mechanics*, volume 191 of *Boston Studies in the Philosophy of Science*. Kluwer Academic Publishers Group, Dordrecht, 1997. Translated from the 1811 French original, with an introduction and edited by Auguste Boissonnade and Vickor N. Vagliente, With a preface by Craig G. Fraser.

[46] Joseph-Louis Lagrange. Théorie des variations périodiques des mouvements des planètes, première et seconde parties. In *Oeuvres de Lagrange*, volume V of *Nouveaux memoires de l'Academie royale des Sciences et Belles-Lettres de Berlin*, pages 347–377 and 417–489. Gauthier-Villars, Paris, 1808.

[47] Serge Lang. *Introduction to differentiable manifolds.* Interscience Publishers (a division of John Wiley & Sons, Inc.), New York-London, 1962.

[48] Serge Lang. *Differential and Riemannian manifolds*, volume 160 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, third edition, 1995.

[49] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics.* Springer, New York, second edition, 2013.

[50] Sophus Lie. *Theory of transformation groups. I.* Springer, Heidelberg, 2015. General properties of continuous transformation groups, A contemporary approach and translation, With the collaboration of Friedrich Engel, Edited and translated from the German and with a foreword by Joël Merker.

[51] Kefeng Liu. Remarks on nonabelian localization. *Internat. Math. Res. Notices*, (13):683–691, 1995.

[52] Kefeng Liu. Heat kernel and moduli space. *Math. Res. Lett.*, 3(6):743–762, 1996.

[53] Kefeng Liu. Heat kernel and moduli spaces. II. *Math. Res. Lett.*, 4(4):569–588, 1997.

[54] I. G. Macdonald. The volume of a compact Lie group. *Invent. Math.*, 56(2):93–95, 1980.

[55] Thomas Bruun Madsen and Andrew Swann. Multi-moment maps. *Adv. Math.*, 229(4):2287–2309, 2012.

[56] Franco Magri. A simple model of the integrable Hamiltonian equation. *J. Math. Phys.*, 19(5):1156–1162, 1978.

[57] M. S. Marinov. Invariant volumes of compact groups. *J. Phys. A*, 13(11):3357–3366, 1980.

[58] Charles-Michel Marle. The inception of symplectic geometry: the works of Lagrange and Poisson during the years 1808–1810. *Lett. Math. Phys.*, 90(1-3):3–21, 2009.

[59] Jerrold Marsden and Alan Weinstein. Reduction of symplectic manifolds with symmetry. *Rep. Mathematical Phys.*, 5(1):121–130, 1974.

[60] Jerrold E. Marsden. *Lectures on mechanics*, volume 174 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1992.

[61] Jerrold E. Marsden, George W. Patrick, and Steve Shkoller. Multisymplectic geometry, variational integrators, and nonlinear PDEs. *Comm. Math. Phys.*, 199(2):351–395, 1998.

[62] Jerrold E. Marsden and Tudor S. Ratiu. *Introduction to mechanics and symmetry*, volume 17 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 1999. A basic exposition of classical mechanical systems.

[63] Jerrold E. Marsden and Tudor S. Ratiu, editors. *The breadth of symplectic and Poisson geometry*, volume 232 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2005. Festschrift in honor of Alan Weinstein.

[64] Jean Martinet. Sur les singularités des formes différentielles. *Ann. Inst. Fourier (Grenoble)*, 20(fasc. 1):95–178, 1970.

[65] Eckhard Meinrenken. Symplectic geometry. Lecture notes, University of Toronto, 2000.

[66] Eckhard Meinrenken. Lectures on group-valued moment maps and Verlinde formulas. In *Mathematical aspects of quantization*, volume 583 of *Contemp. Math.*, pages 175–210. Amer. Math. Soc., Providence, RI, 2012.

[67] Kenneth R. Meyer. Symmetries and integrals in mechanics. pages 259–272, 1973.

[68] John Moore. Lecture notes on Sieberg-Witten invariants. july 2010.

[69] Andrei Moroianu. *Lectures on Kähler geometry*, volume 69 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2007.

[70] Motohico Mulase and Michael Penkava. Volume of representation varieties. preprint, https://arxiv.org/pdf/math/0212012.pdf, December 2002.

[71] Arturo Echeverria-E nriquez, Miguel C. Mu noz Lecanda, and Narciso Roman-Roy. Remarks on multisymplectic reduction. preprint, https://arxiv.org/pdf/1712.09901.pdf, January 2018.

[72] Juan-Pablo Ortega and Tudor S. Ratiu. *Momentum maps and Hamiltonian reduction*, volume 222 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2004.

[73] T. R. Ramadas, I. M. Singer, and J. Weitsman. Some comments on Chern-Simons gauge theory. *Comm. Math. Phys.*, 126(2):409–420, 1989.

[74] Leonid Ryvkin and Tilmann Wurzbacher. An invitation to multisymplectic geometry. preprint, https://arxiv.org/pdf/1804.02553.pdf, April 2018.

[75] Peter Scheinost and Martin Schottenloher. Metaplectic quantization of the moduli spaces of flat and parabolic bundles. *J. Reine Angew. Math.*, 466:145–219, 1995.

[76] M. Schottenloher. *A mathematical introduction to conformal field theory*, volume 759 of *Lecture Notes in Physics*. Springer-Verlag, Berlin, second edition, 2008.

[77] N. Seiberg and E. Witten. Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory. *Nuclear Phys. B*, 426(1):19–52, 1994.

[78] Ambar Sengupta. Sewing symplectic volumes for flat connections over compact surfaces. *J. Geom. Phys.*, 32(3):269–292, 2000.

[79] Ambar N. Sengupta. The volume measure for flat connections as limit of the Yang-Mills measure. *J. Geom. Phys.*, 47(4):398–426, 2003.

[80] Reyer Sjamaar and Eugene Lerman. Stratified symplectic spaces and reduction. *Ann. of Math. (2)*, 134(2):375–422, 1991.

[81] S. Smale. Topology and mechanics. I. *Invent. Math.*, 10:305–331, 1970.

[82] S. Smale. Topology and mechanics. II. The planar n-body problem. *Invent. Math.*, 11:45–64, 1970.

[83] J.-M. Souriau. Quantification géométrique. *Comm. Math. Phys.*, 1:374–398, 1966.

[84] J.-M. Souriau. *Structure des systèmes dynamiques.* Mai trises de mathématiques. Dunod, Paris, 1970.

[85] Edwin H. Spanier. *Algebraic topology.* Springer-Verlag, New York-Berlin, 1981. Corrected reprint.

[86] Clifford Henry Taubes. The Seiberg-Witten and Gromov invariants. *Math. Res. Lett.*, 2(2):221–238, 1995.

[87] Jörg Teschner. Quantization of moduli spaces of flat connections and Liouville theory. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. III*, pages 1223–1246. Kyung Moon Sa, Seoul, 2014.

[88] Michael Thaddeus. Conformal field theory and the cohomology of the moduli space of stable bundles. *J. Differential Geom.*, 35(1):131–149, 1992.

[89] René Thom. Quelques propriétés globales des variétés différentiables. *Comment. Math. Helv.*, 28:17–86, 1954.

[90] V. S. Varadarajan. Vector bundles and connections in physics and mathematics: some historical remarks. In *A tribute to C. S. Seshadri (Chennai, 2002)*, Trends Math., pages 502–541. Birkhäuser, Basel, 2003.

[91] Mikhail Verbitsky. Hyper-Kähler embeddings and holomorphic symplectic geometry. I. *J. Algebraic Geom.*, 5(3):401–413, 1996.

[92] Zhenghan Wang. *Topological quantum computation*, volume 112 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2010.

[93] Alan Weinstein. *Lectures on symplectic manifolds.* American Mathematical Society, Providence, R.I., 1977. Expository lectures from the CBMS Regional Conference held at the University of North Carolina, March 8–12, 1976, Regional Conference Series in Mathematics, No. 29.

[94] Alan Weinstein. Symplectic geometry. *Bull. Amer. Math. Soc. (N.S.)*, 5(1):1–13, 1981.

[95] Alan Weinstein. Sophus Lie and symplectic geometry. *Exposition. Math.*, 1(1):95–96, 1983.

[96] Raymond O. Wells, Jr. *Differential analysis on complex manifolds*, volume 65 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2008. With a new appendix by Oscar Garcia-Prada.

[97] Hermann Weyl. *The classical groups.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1997. Their invariants and representations, Fifteenth printing, Princeton Paperbacks.

[98] Edward Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121(3):351–399, 1989.

[99] Edward Witten. On quantum gauge theories in two dimensions. *Comm. Math. Phys.*, 141(1):153–209, 1991.

[100] Edward Witten. Two-dimensional gauge theories revisited. *J. Geom. Phys.*, 9(4):303–368, 1992.

[101] Edward Witten. Monopoles and four-manifolds. *Math. Res. Lett.*, 1(6):769–796, 1994.

[102] Edward Witten. Mirror symmetry, Hitchin's equations, and Langlands duality. In *The many facets of geometry*, pages 113–128. Oxford Univ. Press, Oxford, 2010.